



ELSEVIER

Contents lists available at ScienceDirect

Journal of Functional Analysis

www.elsevier.com/locate/jfa



On vanishing near corners of transmission eigenfunctions

Eemeli Blåsten^a, Hongyu Liu^{b,c,*}^a Jockey Club Institute for Advanced Study, Hong Kong University of Science and Technology, Hong Kong SAR^b Department of Mathematics, Hong Kong Baptist University, Kowloon Tong, Hong Kong SAR^c HKBU Institute of Research and Continuing Education, Virtual University Park, Shenzhen, PR China

ARTICLE INFO

Article history:

Received 31 January 2017

Accepted 29 August 2017

Available online 5 September 2017

Communicated by M. Hairer

MSC:

35P25

58J50

35R30

81V80

Keywords:

Interior transmission eigenfunction

Corner

Vanishing and localizing

Non-scattering

ABSTRACT

Let Ω be a bounded domain in \mathbb{R}^n , $n \geq 2$, and $V \in L^\infty(\Omega)$ be a potential function. Consider the following transmission eigenvalue problem for nontrivial $v, w \in L^2(\Omega)$ and $k \in \mathbb{R}_+$,

$$\begin{cases} (\Delta + k^2)v = 0 & \text{in } \Omega, \\ (\Delta + k^2(1 + V))w = 0 & \text{in } \Omega, \\ w - v \in H_0^2(\Omega), \quad \|v\|_{L^2(\Omega)} = 1. \end{cases}$$

We show that the transmission eigenfunctions v and w carry the geometric information of $\text{supp}(V)$. Indeed, it is proved that v and w vanish near a corner point on $\partial\Omega$ in a generic situation where the corner possesses an interior angle less than π and the potential function V does not vanish at the corner point. This is the first quantitative result concerning the intrinsic property of transmission eigenfunctions and enriches the classical spectral theory for Dirichlet/Neumann Laplacian. We also discuss its implications to inverse scattering theory and invisibility.

© 2017 Elsevier Inc. All rights reserved.

* Corresponding author.

E-mail addresses: iaseemeli@ust.hk (E. Blåsten), hongyu.liuip@gmail.com (H. Liu).

1. Introduction

Let Ω be a bounded domain in \mathbb{R}^n , $n \geq 2$, and $V \in L^\infty(\Omega)$ be a potential function. Consider the following (interior) transmission eigenvalue problem for $v, w \in L^2(\Omega)$,

$$\begin{cases} (\Delta + k^2)v = 0 & \text{in } \Omega, \\ (\Delta + k^2(1 + V))w = 0 & \text{in } \Omega, \\ w - v \in H_0^2(\Omega), \quad \|v\|_{L^2(\Omega)} = 1. \end{cases} \quad (1.1)$$

If the system (1.1) admits a pair of nontrivial solutions (v, w) , then k is referred to as an (*interior*) *transmission eigenvalue* and (v, w) is the corresponding pair of (*interior*) *transmission eigenfunctions*. Note in particular that nothing is imposed a-priori on the boundary values of v or w individually. In this paper, we are mainly interested in the real eigenvalues, $k \in \mathbb{R}_+$, which are physically relevant. The study of the transmission eigenvalue problem has a long history and is of significant importance in scattering theory. The transmission eigenvalue problem is a type of non elliptic and non self-adjoint problem, so its study is mathematically interesting and challenging. In the literature, the existing results are mainly concerned with the spectral properties of the transmission eigenvalues, including the existence, discreteness and infiniteness, and Weyl laws; see for example [4,7,11,25,30–32] and the recent survey [8]. There are few results concerning the intrinsic properties of the transmission eigenfunctions. Here we are aware that the completeness of the set of *generalized transmission eigenfunctions* in L^2 is proven in [4,31].

In this paper, we are concerned with the vanishing properties of *interior transmission eigenfunctions*. It is shown that in admissible geometric situations, transmission eigenfunctions which can be approximated suitably by Herglotz waves will vanish at corners of the support of the potential V . To our best knowledge, this is the first quantitative result on intrinsic properties of transmission eigenfunctions. As expected, these carry geometric information of the support of the underlying potential V as well as other interesting consequences and implications in scattering theory, which we shall discuss in more details in Section 7.

The location of vanishing of eigenfunctions is an important area of study in the classical spectral theory for the Dirichlet/Neumann Laplacian. Two important topics are the *nodal sets* and *eigenfunction localization*. The former is the set of points in the domain where the eigenfunction vanishes. For the latter, an eigenfunction is said to be *localized* if most of its L^2 -energy is contained in a subdomain which is a fraction of the total domain. Considerable effort has been spent on the nodal sets and localization in the classical spectral theory. We refer to the recent survey [17]. For the curious, we mention briefly basic facts about them, all of which are completely open for transmission eigenfunctions. Nodal sets are C^∞ -curves whose intersections form equal angles. By the celebrated Courant's nodal line theorem, the nodal set of the m -th eigenfunction divides the domain into at most m nodal domains. Localization seems to be a more recent topic

even though some examples have been known for a long time. A such example is the *whispering gallery modes* that comes from Lord Rayleigh's study of whispering waves in the Saint Paul Cathedral in London during the late 19th century. These eigenfunctions concentrate their energy near the boundary of a spherical or elliptical domain. Other well known localized modes are called *bouncing ball modes* and *focusing modes* [9,23]. It is worth noting that the Laplacian does not possess localized eigenfunctions on rectangular or equilateral triangular domains [28]. However, localization does appear for the classical eigenvalue problem in a certain sense when the angle is reflex [27]. We also refer to [20] for more relevant examples.

In our case of the transmission eigenvalue problem, peculiar and intriguing phenomena are observed in that both vanishing and localization of transmission eigenfunctions may occur near corners of the support of the potential. Indeed, in an upcoming numerical paper [3], we show that if the interior angle of a corner is less than π , then the transmission eigenfunctions vanish near the corner, whereas if the interior angle is bigger than π , then the transmission eigenfunctions localize near the corner. In this paper, we shall rigorously justify the vanishing property of the transmission eigenfunction in a certain generic situation. It turns out to be a highly technical matter. In fact, even in the classical spectral theory, the intrinsic properties of the eigenfunctions are much more difficult to study than those of the eigenvalues, and they remain a fascinating topic for a lot of ongoing research. Nevertheless, we would also like to mention that with the help of highly accurate computational methods, we can present a more detailed numerical investigation in [3] including the vanishing/localizing order as well as its relationship to the angle of the corner.

We believe that the vanishing and localizing properties of transmission eigenfunctions are closely related to the analytic continuation of the eigenfunctions. Indeed in the recent papers [5,14,15,21,29], it is shown that transmission eigenfunctions cannot be extended analytically to a neighborhood of a corner. The failure of the analytic continuation of transmission eigenfunctions can be used via an absurdity argument in [21] to show the uniqueness in determining the polyhedral support of an inhomogeneous medium by a single far-field pattern in the inverse scattering theory. By further quantifying the aforementioned analytic continuation property of transmission eigenfunctions, sharp stability estimates were established in [1] in determining the polyhedral support of an inhomogeneous medium by a single far-field pattern. Those uniqueness and stability results already indicate that the intrinsic properties of transmission eigenfunctions carry geometric information of the underlying potential function V . Furthermore in [1], as an interesting consequence of the quantitative estimates involved, a sharp lower bound can be derived for the far-field patterns of the waves scattered from polyhedral potentials associated with incident plane waves. In this paper, we can significantly extend this result by establishing a similar quantitative lower bound associated with incident Herglotz waves. On the other hand, it is known [6] that the scattered waves created by incident waves that are Herglotz approximations to transmission eigenfunctions will have an arbitrarily small far-field energy. This critical observation apparently indicates that the

transmission eigenfunctions must vanish near the corner point. We shall give more relevant discussion of our results in Section 7, connecting our study to inverse scattering problems and invisibility cloaking.

The rest of the paper is organized as follows. We will recall scattering theory and define notation in Section 2. All of the background and admissibility assumptions are contained therein. We state our main results mathematically in Section 3, and then proceed to prove them in Section 5 and Section 6 using results from Section 4.

2. Preliminaries

In this section we recall background theory, lay some definitions and fix notation. We will start by describing acoustic scattering theory for penetrable scatterers. This will be referred to as “background assumptions” in theorems. After that we recall what is the interior transmission problem and some of its known facts. Finally we define which potentials are admissible for our theorems.

2.1. Background assumptions

Whenever we say that “let the background assumptions hold” we mean that everything in this section should hold, unless stated otherwise. We will recall the fundamentals of acoustic scattering theory. For more details in the three dimensional case we refer the readers to [12].

We will consider only scatterers of finite diameter that are contained in a large origin-centered ball, the *domain of interest*,

$$B_R = B(\bar{0}, R) = \{x \in \mathbb{R}^n \mid |x| < R\}$$

where $R > 1$ is fixed. Let $V \in L^\infty(B_R)$ be a bounded potential function representing the medium parameter of the scatterer. We shall consider scattering of a fixed frequency by fixing the wavenumber $k \in \mathbb{R}_+$.

The scatterer V is illuminated by an incident wave, which in this paper is chosen to be any Herglotz wave. These are superpositions of plane-waves that can be written as

$$u^i(x) = \int_{\mathbb{S}^{n-1}} e^{ik\theta \cdot x} g(\theta) d\sigma(\theta) \quad (2.1)$$

where the kernel $g \in L^2(\mathbb{S}^{n-1})$. We say that u^i is *normalized* if $\|g\|_{L^2(\mathbb{S}^{n-1})} = 1$. The field u^i is called *incident* because it satisfies the equation

$$(\Delta + k^2)u^i = 0$$

which corresponds to a background unperturbed by the presence of V .

Unless V is transparent to u^i , the illumination of V by u^i creates a unique scattered wave $u^s \in H^2_{\text{loc}}(\mathbb{R}^n)$ such that

$$\begin{aligned} (\Delta + k^2(1 + V))u &= 0 \quad \text{in } \mathbb{R}^n, \\ u &= u^i + u^s, \\ \lim_{r \rightarrow \infty} r^{\frac{n-1}{2}} (\partial_r u^s - ik u^s) &= 0. \end{aligned} \tag{2.2}$$

Here u is the *total field* which, as a superposition of the incident field and scattered field, represents the physical observable field. The third condition, where $r = |x|$, says that u^s satisfies the *Sommerfeld radiation condition*, which can be interpreted as having u^s propagating from V to infinity instead of the other way around.

A property of the scattered field is that as one zooms out, the potential V starts to look more and more like a point-source in a sense. This means that far away, u^s looks like the Green’s function to $\Delta + k^2$ but modulated by a *far-field pattern* u^s_∞ . More precisely, as $|x| \rightarrow \infty$, u has the expansion

$$u(x) = u^i(x) + \frac{e^{ik|x|}}{|x|^{(n-1)/2}} u^s_\infty \left(\frac{x}{|x|}; u^i \right) + \mathcal{O} \left(\frac{1}{|x|^{n/2}} \right)$$

where for a fixed u^i the far-field pattern is a real-analytic map $u^s_\infty : \mathbb{S}^{n-1} \rightarrow \mathbb{C}$ (it is also called the *scattering amplitude*).

2.2. The interior transmission problem

Direct scattering theory is all about the study of the map $(u^i, V) \mapsto u^s_\infty$. Given a potential V the *far-field operator*¹ maps the Herglotz kernel g of u^i to the far-field pattern u^s_∞ . In inverse scattering one is interested in recovering meaningful information about the scatterer V from full or partial information of the far-field operator.

A number of algorithms in inverse scattering, such as linear sampling [10] and factorization methods [24] fail at wavenumbers where the far-field operator has non-trivial kernel. In such a case there is an incident wave u^i for which V does not cause a detectable change in the far-field, and thus by Rellich’s lemma and unique continuation u^i does not scatter at all: $\text{supp } u^s \subset \overline{\Omega}$. If this happens we call k a *non-scattering energy* (or wavenumber) and say that V is transparent to u^i , or that u^i is non-scattering. It is known that there are radially symmetric potentials which are transparent to certain incident waves [16].

If u^i is non-scattering and we restrict it to the supporting set Ω , then the following *interior transmission problem* has a non-trivial solution $(v, w) \in L^2(\Omega) \times L^2(\Omega)$

¹ Also called the *relative scattering operator*. The unitary *scattering operator* is the identity plus the former.

$$(\Delta + k^2)v = 0 \quad \text{in } \Omega, \tag{2.3}$$

$$(\Delta + k^2(1 + V))w = 0 \quad \text{in } \Omega, \tag{2.4}$$

$$w - v \in H_0^2(\Omega), \tag{2.5}$$

namely $v = u_{|\Omega}^i$ and $w = u_{|\Omega}$. When this non-elliptic, non self-adjoint eigenvalue problem has a solution, we call k a transmission eigenvalue. The functions v and w are referred to as the transmission eigenfunctions.

If V is radially symmetric, then v in (2.3) extends to the whole \mathbb{R}^n as a Herglotz function, and hence in this case transmission eigenvalues and non-scattering energies coincide [13]. This observation hinted for a long time that these sets of numbers coincide in general. However it was a red herring: a series of papers on corner scattering [5,14,15,29] showed that in the presence of a certain type of corner or edge singularity in the potential V there are no non-scattering energies despite the well-known fact that such a scatterer always has an infinite discrete set of transmission eigenvalues.

We remark that the problem (2.3)–(2.5) has been studied heavily [8]. Many properties of the transmission eigenvalues are known. Despite this almost nothing is known about the eigenfunctions themselves before this paper.

2.3. Herglotz approximation

We introduced the Herglotz wave function in (2.1), which shall be used to approximate the transmission eigenfunction v satisfying (2.3). We briefly recall the following result concerning the Herglotz approximation for the subsequent use.

Theorem 2.1 (Theorem 2 in [36]). *Let \mathbf{W}_k denote the space of all Herglotz wave functions of the form (2.1). For $\Omega \subset \mathbb{R}^n$ a C^0 -domain, define*

$$\mathbf{U}_k(\Omega) := \{u \in C^\infty(\Omega); (\Delta + k^2)u = 0\},$$

and

$$\mathbf{W}_k(\Omega) := \{u_{|\Omega}; u \in \mathbf{W}_k\}.$$

Then $\mathbf{W}_k(\Omega)$ is dense in $\mathbf{U}_k(\Omega) \cap L^2(\Omega)$ with respect to the topology induced by the L^2 -norm.

2.4. Admissible potentials

As part of our proof of the vanishing of transmission eigenfunctions at corners we will show lower bounds for the far-field pattern u^s . That is, we shall consider the scattering from a corner and make use of the corner singularity in the potential. To save notational burden we collect these a-priori assumptions in this section.

We shall only consider polygonal or hypercuboidal scatterers V for simplicity. In essence V will be defined as a Hölder-continuous function φ restricted to a polygonal domain Ω ; see below. As the arguments are local, the results will hold qualitatively for any potential V for which $V|_{\mathcal{U}} = \chi_{\Omega}\varphi|_{\mathcal{U}}$ for some open set \mathcal{U} and such that there is a reasonable path from \mathcal{U} to infinity.

Definition 2.2. Recalling the notation B_R from Section 2.1, we say that the potential V is (qualitatively) admissible if

- (1) $V = \chi_{\Omega}\varphi$, where $\chi_{\Omega}(x) = 1$ if $x \in \Omega$ and $\chi_{\Omega}(x) = 0$ otherwise;
- (2) $\Omega \subset B_R$ is an open convex polygon in 2D or a cuboid in higher dimensions;
- (3) $\varphi \in C^{\alpha}(\mathbb{R}^n)$ for some $\alpha > 0$ in 2D and $\alpha > 1/4$ in higher dimensions;
- (4) $\varphi \neq 0$ at some vertex of Ω .

2.5. Function order

An important concept in corner scattering is the so-called *function order*. This determines how flat the function is at a certain point, or in other words what is the order of the first non-trivial homogeneous polynomial in its Taylor expansion at that point.

Definition 2.3. Let f be a complex-valued function defined in an open neighborhood of $x_c \in \mathbb{R}^n$. We say that f has order N at x_c if

$$N = \max\{M \in \mathbb{Z} \mid \exists C < \infty : |f(x)| \leq C|x - x_c|^M \text{ near } x_c\}.$$

If the set is unbounded from above we say that f has order ∞ . If the set is empty f has order $-\infty$.

Remark 2.4. If f is smooth then it has order $N < \infty$ at x_c if and only if $\partial^{\alpha}f(x_c) = 0$ for $\alpha \in \mathbb{N}^n$, $|\alpha| < N$ and $\partial^{\beta}f(x_c) \neq 0$ for some $\beta \in \mathbb{N}^n$, $|\beta| = N$. When $N = \infty$ the second condition is ignored: there are smooth functions vanishing to infinite order e.g. $\exp(-1/|x|^2)$. Smooth functions always have non-negative order.

3. Statement of the main results

Theorem 3.1. *Let $n \in \{2, 3\}$ and let the background assumptions hold. If V is qualitatively admissible with $\varphi(x_c) \neq 0$ at a vertex x_c of Ω , and $\mathcal{N} \in \mathbb{N}$, then there is $c, \ell < \infty$ depending on V, n, k, \mathcal{N} and $\mathcal{S} = \mathcal{S}(V, k) \geq 1$ such that*

$$\|u_{\infty}^{\mathcal{S}}\|_{L^2(\mathbb{S}^{n-1})} \geq \frac{\mathcal{S}}{\exp \exp (c \min(1, \|P_N\|)^{-\ell})} \tag{3.1}$$

for any normalized incident Herglotz wave u^i which is of order $N \leq \mathcal{N}$ at x_c and whose Taylor expansion there begins with P_N . Here $\|P_N\| = \int_{\mathbb{S}^{n-1}} |P_N(\theta)| d\sigma(\theta)$.

Theorem 3.2. *Let $n \in \{2, 3\}$ and V be a qualitatively admissible potential. Assume that $k > 0$ is a transmission eigenvalue: there exists $v, w \in L^2(\Omega)$ such that*

$$\begin{aligned} (\Delta + k^2)v &= 0 \quad \text{in } \Omega \\ (\Delta + k^2(1 + V))w &= 0 \quad \text{in } \Omega \\ w - v &\in H_0^2(\Omega), \quad \|v\|_{L^2(\Omega)} = 1. \end{aligned}$$

If v can be approximated in the $L^2(\Omega)$ -norm by a sequence of Herglotz waves with uniformly $L^2(\mathbb{S}^{n-1})$ -bounded kernels, then

$$\lim_{r \rightarrow 0} \frac{1}{m(B(x_c, r))} \int_{B(x_c, r)} |v(x)| \, dx = 0$$

where x_c is any vertex of Ω such that $\varphi(x_c) \neq 0$.

Remark 3.3. A sequence of Herglotz waves v_j with uniformly bounded kernels has uniformly bounded L^2 -norms in any fixed bounded set. However the converse is not true by inspecting a sequence of spherical harmonics $g_j = Y_j^0$. In other words the condition we have here is rather technical. See Section 7 for more relevant discussion.

4. Auxiliary results

In this section, we collect three auxiliary propositions that follow without too much effort from our previous results in [1] concerning the corner scattering. We add a proposition showing that in the presence of transmission eigenfunctions incident waves creating arbitrary small far-field patterns can be generated. Finally, another proposition gives a lower bound for the Laplace transform of a harmonic polynomial. The latter is necessary for quantitative estimates involving incident Herglotz waves in corner scattering. In comparison, we note that the paper [1] is mainly concerned with corner scattering associated with incident plane waves.

Proposition 4.1. *Let the background assumptions hold with $n \in \{2, 3\}$, V qualitatively admissible, u^i a normalized Herglotz wave and let $\mathcal{S} \geq 1$. Then there is $\varepsilon_m(\mathcal{S}, k, R) > 0$ such that if $\|u^s\|_{H^2(B_{2R})} \leq \mathcal{S}$ and $\|u_\infty^s\|_{L^2(\mathbb{S}^{n-1})} \leq \varepsilon_m$ then*

$$\sup_{x \in \partial\Omega} |u^s(x)| + |\nabla u^s(x)| \leq \frac{c}{\sqrt{\ln \ln \frac{\mathcal{S}}{\|u_\infty^s\|_{L^2(\mathbb{S}^{n-1})}}}} \tag{4.1}$$

for some $c = c(V, \mathcal{S}, k, R) < \infty$.

This is a less general version of Proposition 5.10 in our previous paper. We will also need a “converse” result estimating the far-field pattern by the near-field. In more detail,

we will build incident waves with arbitrarily small far-field patterns in the presence of a transmission eigenfunction (cf. [6]).

Proposition 4.2. *Let the background assumptions hold with V supported in $\overline{\Omega}$, and assume that $(v, w) \in L^2(\Omega) \times L^2(\Omega)$ are a pair of transmission eigenfunctions on a bounded domain Ω . There is $C = C(V, k) < \infty$ such that if $v_j \in L^2_{loc}$ is an incident wave such that $\|v - v_j\|_{L^2(\Omega)} < \varepsilon$ then the produced far-field pattern has $\|v^s_{j\infty}\|_{L^2(\mathbb{S}^{n-1})} < C\varepsilon$.*

Proof. Let v^i_0 be the zero-extension of v to the whole \mathbb{R}^n , and let v^s_0 be the radiating solution to $(\Delta + k^2(1 + V))v^s_0 = -k^2Vv^i_0$. Also let v^s_0 be the zero-extension of $w - v \in H^2_0(\Omega)$ to \mathbb{R}^n . By standard scattering theory (e.g. Chapter 8 in [12]) we see that $v^s_0 = v^s_0$ since

$$(\Delta + k^2(1 + V))v^s_0 = -k^2Vv^i_0 = -k^2Vv = (\Delta + k^2(1 + V))v^s_0$$

in \mathbb{R}^n and both satisfy the Sommerfeld radiation condition trivially. Hence the far-field pattern of v^s_0 is zero.

Since v_j approximates v in $L^2(\Omega)$, and V is supported on $\overline{\Omega}$, we have $-k^2Vv_j$ approximating $-k^2Vv^i_0$ in \mathbb{R}^n . Let v^s_j be the scattered wave arising from the incident wave v_j and potential V . Then, again from standard scattering theory, its far-field pattern approximates the far-field pattern of v^s_0 , i.e. zero. The operators involved are all bounded, so

$$\|v^s_{j\infty}\|_{L^2(\mathbb{S}^{n-1})} < C_{V,k}\varepsilon. \quad \square \tag{4.2}$$

We also recall the existence of complex geometrical optics solutions.

Proposition 4.3. *Let $n \in \{2, 3\}$, $k > 0$ and let V be a qualitatively admissible potential. Then there is $p = p(V, n) \geq 2$ and $c = c(V, R, k, n) < \infty$ with the following properties: if $\rho \in \mathbb{C}^n$ satisfies $\rho \cdot \rho + k^2 = 0$ and $|\Im \rho| \geq c^{(n+1)/2}$ then there is $\psi \in L^p(\mathbb{R}^n)$ such that $u_0(x) = e^{\rho \cdot x}(1 + \psi(x))$ solves $(\Delta + k^2(1 + V))u_0 = 0$ in \mathbb{R}^n , and*

$$\|\psi\|_{L^p(\mathbb{R}^n)} \leq c |\Im \rho|^{-n/p-\beta}$$

for some $\beta = \beta(V, n) > 0$. In addition there is the norm estimate $\|\psi\|_{H^2(B_{2R})} \leq c|\rho|^2$.

Proposition 4.3 specializes Proposition 7.6 from [1]. Also, mainly by Corollary 6.2 from that same paper, together with the use of Taylor’s theorem on the real-analytic incident wave u^i , we can show

Proposition 4.4. *Let $n \in \{2, 3\}$ and let the background assumptions hold with u^i a normalized Herglotz wave. Let $V = \chi_\Omega \varphi$ be a qualitatively admissible potential. Choose*

coordinates such that the origin is a vertex of Ω where $\varphi \neq 0$. Let $N \in \mathbb{N}$ be such that $\partial^\gamma u^i(\bar{0}) = 0$ for $|\gamma| < N$ and set

$$P_N(x) = \sum_{|\gamma|=N} \frac{\partial^\gamma u^i(\bar{0})}{\gamma!} x^\gamma.$$

Let $\rho \in \mathbb{C}^n$ be such that it satisfies the assumptions of Proposition 4.3, $|\Re\rho| \geq \max(1, k)$ and $\Re\rho \cdot x \leq -\delta_0 |x| |\Re\rho|$ for some $\delta_0 > 0$ and any $x \in \Omega$. Then

$$c \left| \int_{\mathfrak{C}} e^{\rho \cdot x} P_N(x) dx \right| \leq |\Re\rho|^{-N-n-\min(1,\alpha,\beta)} + |\Re\rho|^3 \sup_{\partial(\mathfrak{C} \cap B(\bar{0}, h))} \{|u^s|, |\nabla u^s|\} \quad (4.3)$$

where \mathfrak{C} is the open cone generated by Ω at the origin, $h = h(\Omega)$ is the minimal distance from any vertex of Ω to any of its non-adjacent edges, and the constant $c > 0$ depends on V, N, δ_0 and k .

Next is the turn of a lower bound to the Laplace transform for homogeneous harmonic polynomials of arbitrary degree. The proof is a compactness argument with basis in the non-vanishingness proofs from [5] and [29]. We recall that the norm for homogeneous polynomials is

$$\|P\| = \int_{\mathbb{S}^{n-1}} |P(\theta)| d\sigma(\theta).$$

Proposition 4.5. Let $n \in \{2, 3\}$, $\mathfrak{C} \neq \emptyset$ be either an open orthant (3D) or an oblique open cone (2D). For $N \in \mathbb{N}$ set

$$\mathcal{P}_N = \left\{ P : \mathbb{C}^n \rightarrow \mathbb{C} \mid \Delta P \equiv 0, P(x) = \sum_{|\gamma|=N} c_\gamma x^\gamma \right\}.$$

Let the angle of \mathfrak{C} be at most $2\alpha_m < \pi$ and let $\alpha_m + \alpha_d < \pi/2$. Then there is $\tau_0 > 0$ and $c > 0$, both depending only on $\mathfrak{C}, N, n, \alpha_m + \alpha_d$ with the following properties: If $P \in \mathcal{P}_N$ then there is a curve $\tau \mapsto \rho(\tau) \in \mathbb{C}^n$ satisfying $\rho(\tau) \cdot \rho(\tau) + k^2 = 0$, $\tau = |\Re\rho(\tau)|$,

$$\Re\rho(\tau) \cdot x \leq -\cos(\alpha_m + \alpha_d) |\Re\rho(\tau)| |x|$$

for all $x \in \mathfrak{C}$, and such that if $\tau \geq \tau_0$ then

$$\left| \int_{\mathfrak{C}} e^{\rho(\tau) \cdot x} P(x) dx \right| \geq \frac{c \|P\|}{|\Re\rho(\tau)|^{N+n}}. \quad (4.4)$$

Proof. We identify \mathcal{P}_N with a subset of \mathbb{C}^m , where $m = \#\{\gamma \in \mathbb{N}^n \mid |\gamma| = N\} = (N + n - 1)! / (N!(n - 1)!)$, by mapping $P \in \mathcal{P}_N$ to the point corresponding to its coefficients listed in some fixed order (e.g. by the lexical order of the multi-indices γ). This induces a topology on \mathcal{P}_N which makes it a complete metric space. The space $\mathcal{P}_N \cap \{\|P\| = 1\}$ is compact.

We will first consider the easier case of a complex vector satisfying $\zeta \cdot \zeta = 0$ instead of $\rho \cdot \rho + k^2 = 0$. Write $\delta_0 = \cos(\alpha_m + \alpha_d)$ and set

$$R_{\mathfrak{C}, \delta_0} = \{\zeta \in \mathbb{C}^n \mid \zeta \cdot \zeta = 0, |\Re \zeta| = 1, \Re \zeta \cdot x \leq -\delta_0 |\Re \zeta| |x| \forall x \in \mathfrak{C}\}.$$

Also, write $\mathcal{L}P(\zeta) = \int_{\mathfrak{C}} \exp(\zeta \cdot x) P(x) dx$ for $P \in \mathcal{P}_N$ and $\zeta \in R_{\mathfrak{C}, \delta_0}$. We claim first that

$$\inf_{P \in \mathcal{P}_N} \sup_{\zeta \in R_{\mathfrak{C}, \delta_0}} |\mathcal{L}P(\zeta)| = c \|P\| \tag{4.5}$$

for some constant $c = c(N, \mathfrak{C}, \delta_0) > 0$. By dividing P with $\|P\|$ and the linearity of \mathcal{L} we may assume that $\|P\| = 1$. If (4.5) did not hold then for any $j \in \mathbb{N}$ there is $P_j \in \mathcal{P}_N, \|P_j\| = 1$ such that $|\mathcal{L}P_j(\zeta)| < j^{-1}$ for any $\zeta \in R_{\mathfrak{C}, \delta_0}$. Since $\mathcal{P}_N \cap \{\|P\| = 1\}$ is compact there is $P_\infty \in \mathcal{P}_N, \|P_\infty\| = 1$ and a subsequence $P_{j_\ell} \rightarrow P_\infty$. Let $\zeta \in R_{\mathfrak{C}, \delta_0}$. It is easily seen that $|\mathcal{L}(P_{j_\ell} - P_\infty)(\zeta)| \leq (N + n - 1)! \delta_0^{1-N-n} \|P_{j_\ell} - P_\infty\| \rightarrow 0$ as $\ell \rightarrow \infty$. Hence $|\mathcal{L}P_\infty(\zeta)| = 0$ for any complex vector $\zeta \in R_{\mathfrak{C}, \delta_0}$, but this contradicts the Laplace transform lower bounds from [5] and [29]. Thus the lower bound (4.5) holds, but for vectors satisfying $\zeta \cdot \zeta = 0$.

Let us build $\rho(\tau)$ by using a ζ from the previous paragraph. Let $P \in \mathcal{P}_N$ be arbitrary and take $\zeta \in R_{\mathfrak{C}, \delta_0}$ such that $|\mathcal{L}P(\zeta)| \geq c \|P\| / 2$. For $\tau > 0$ set

$$\rho(\tau) = \tau \Re \zeta + i \sqrt{\tau^2 + k^2} \Im \zeta.$$

Then $\rho(\tau) / \tau \rightarrow \zeta$ as $\tau \rightarrow \infty$ and moreover $\rho(\tau) \cdot \rho(\tau) + k^2 = 0$, and $\Re \rho(\tau) \cdot x \leq -\delta_0 |\Re \rho(\tau)| |x|$ for $x \in \mathfrak{C}$. When τ is large enough we will have $|\mathcal{L}P(\rho(\tau) / \tau)| \geq c \|P\| / 4$. The proof is as follows: set

$$f(r) = \exp((\Re \zeta + ir \Im \zeta) \cdot x).$$

Then $f(1) = \exp(\zeta \cdot x)$ and $f\left(\sqrt{1 + k^2 / \tau^2}\right) = \exp(\rho(\tau) \cdot x / \tau)$. By the mean value theorem

$$\left| f(1) - f\left(\sqrt{1 + k^2 / \tau^2}\right) \right| \leq \sup_{1 < r < \sqrt{1 + k^2 / \tau^2}} |f'(r)| \left| \sqrt{1 + k^2 / \tau^2} - 1 \right|.$$

But note that $\sqrt{1 + k^2 / \tau^2} - 1 = \tau^{-1} k^2 / (\tau + \sqrt{\tau^2 + k^2}) \leq k / \tau$. Also $f'(r) = i \Im \zeta \cdot x f(r)$ and since $|\Re \zeta| = |\Im \zeta| = 1$ we get $|f'(r)| \leq |x| \exp(-\delta_0 |x|)$. In other words

$$\left| f(1) - f\left(\sqrt{1 + k^2 / \tau^2}\right) \right| \leq \frac{k}{\tau} |x| e^{-\delta_0 |x|}.$$

Finally we see the claim:

$$\begin{aligned} \left| \mathcal{L}P(\zeta) - \mathcal{L}P\left(\frac{\rho(\tau)}{\tau}\right) \right| &= \left| \int_{\mathfrak{e}} \left(f(1) - f\left(\sqrt{1 + k^2/\tau^2}\right) \right) P(x) dx \right| \\ &\leq \frac{k}{\tau} \int_{\mathfrak{e}} e^{-\delta_0|x|} |x| |P(x)| dx = \|P\| \frac{k}{\tau} \int_0^\infty e^{-\delta_0 r} r^{1+N+n-1} dr \\ &= (N+n)! \delta_0^{-N-n} k \tau^{-1} \|P\|, \end{aligned}$$

and so $|\mathcal{L}P(\rho(\tau)/\tau)| > c \|P\| / 4$ if $\tau > 4(N+n)! \delta_0^{-N-n} k / c$. A change of variables gives then $\mathcal{L}P(\rho(\tau)/\tau) = \tau^{N+n} \mathcal{L}P(\rho(\tau))$ and so the proposition is proven. \square

5. Bound for far-field pattern with incident Herglotz wave

Proof of Theorem 3.1. Let $\mathcal{S} = \mathcal{S}(V, k)$ be such that $\|u^s\|_{H^2(B_{2R})} \leq \mathcal{S}$ whenever the incident wave is a normalized Herglotz wave. Let u^i be a normalized incident wave and u^s the corresponding scattered wave. Let u^i be of order $N \in \mathbb{N}$ at the vertex x_c , which we may take as being the origin, and on which $\varphi \neq 0$. Moreover let P_N be its N -th degree homogeneous Taylor polynomial at $\bar{0}$. Note that this polynomial is harmonic because $(\Delta + k^2)u^i = 0$. Firstly combine (4.4), (4.3) and (4.1) to get

$$c \|P_N\| \leq |\Re \rho(\tau)|^{-\min(1, \alpha, \beta)} + \frac{|\Re \rho(\tau)|^{N+n+3}}{\sqrt{\ln \ln \frac{\mathcal{S}}{\|u_\infty^s\|_{L^2(\mathbb{S}^{n-1})}}}}$$

when $\|u_\infty^s\| \leq \varepsilon_m$ and $\tau \geq \tau_0$, with constants depending on $V, N, n, k, \alpha_m + \alpha_d, \mathcal{S}$.

The estimate above depends monotonically on each individual constant. Fix $\mathcal{N} \in \mathbb{N}$ and set

$$\varepsilon_{m, \mathcal{N}} = \min_{N \leq \mathcal{N}} \varepsilon_m, \quad \tau_{0, \mathcal{N}} = \max_{N \leq \mathcal{N}} \tau_0, \quad c_{\mathcal{N}} = \min_{N \leq \mathcal{N}} c.$$

Then if $\mathcal{N} \geq N$ the estimate holds with these new constants and \mathcal{N} in the exponent instead of N (since $|\Re \rho(\tau)| = \tau \geq 1$). In other words

$$c_{\mathcal{N}} \|P_N\| \leq |\Re \rho(\tau)|^{-\min(1, \alpha, \beta)} + \frac{|\Re \rho(\tau)|^{\mathcal{N}+n+3}}{\sqrt{\ln \ln \frac{\mathcal{S}}{\|u_\infty^s\|_{L^2(\mathbb{S}^{n-1})}}}} \tag{5.1}$$

when $\|u_\infty^s\| \leq \varepsilon_{m, \mathcal{N}}$ and $\tau \geq \tau_{0, \mathcal{N}}$ and u^i is of order $N \leq \mathcal{N}$ at $\bar{0}$.

Write $\gamma = \min(1, \alpha, \beta)$ and $R = \sqrt{\ln \ln (\mathcal{S} / \|u_\infty^s\|_{L^2(\mathbb{S}^{n-1})})}$. The right-hand side of (5.1) has a global minimum at the point

$$\tau_m = (\gamma R / (\mathcal{N} + n + 3))^{1/(\mathcal{N} + n + 3 + \gamma)},$$

and the minimal value there is given by $c(\mathcal{N}, n, \gamma)R^{-\gamma/(\mathcal{N}+n+3+\gamma)}$. Hence if $\tau_m \geq \tau_{0,\mathcal{N}}$, we may set $\tau = \tau_m$ in (5.1) and solve for the norm of the far-field pattern. We then have

$$\|u_\infty^s\|_{L^2(\mathbb{S}^{n-1})} \geq \frac{\mathcal{S}}{\exp \exp (c \|P_N\|^{-\ell})} \tag{5.2}$$

where the exponent $\ell \geq 2(\mathcal{N} + n + 4)$ and $c < \infty$ may be chosen to depend only on V, n, k, \mathcal{N} . The other case, namely $\tau_m < \tau_{0,\mathcal{N}}$ reduces to $\|u_\infty^s\|_{L^2(\mathbb{S}^{n-1})} > \mathcal{S}/(\exp \exp c)$ for some $c = c(V, n, k, \mathcal{N})$. \square

6. Vanishing of the interior transmission eigenfunction at corners

Proof of Theorem 3.2. Let us start by taking a sequence of incident Herglotz waves

$$v_j(x) = \int_{\mathbb{S}^{n-1}} \exp(ik\theta \cdot x)g_j(\theta)d\sigma(\theta)$$

approximating the interior transmission eigenfunction v in the $L^2(\Omega)$ -norm; see Theorem 2.1. We may assume for example that $\|v - v_j\|_{L^2(\Omega)} < 2^{-j}$. By Proposition 4.2 we have the estimate

$$\|v_{j\infty}^s\|_{L^2(\mathbb{S}^{n-1})} < C_{V,k}2^{-j} \tag{6.1}$$

for the corresponding far-field pattern. The assumption on v allows us to have $\|g_j\|_{L^2(\mathbb{S}^{n-1})} \leq G < \infty$ for all j .

Let $x_c \in \partial\Omega$ be a vertex such that $\varphi(x_c) \neq 0$. Our goal is to estimate the integral of $|v|$ in $B(x_c, r) \cap \Omega$. We will achieve that by estimating the corresponding integrals of v_j . Let us denote $B = B(x_c, r)$ for convenience. Let N_j be the order of v_j at x_c , so $\partial^\alpha v_j(x_c) = 0$ for $|\alpha| < N_j$. Then by the smoothness of v_j we have $N_j \in \mathbb{N} \cup \{\infty\}$. By its real-analyticity we have $N_j < \infty$. Fix $N \in \mathbb{N}$. If $N_j \geq N$, then

$$\|v\|_{L^1(B \cap \Omega)} \leq \|v - v_j\|_{L^1(B \cap \Omega)} + \|v_j\|_{L^1(B)} \leq C_\Omega 2^{-j} + C_{N,v_j} r^{N+n}.$$

The theorem would follow if $N_j \geq 1$ for an infinite sequence of j 's and $\sup_j C_{N,v_j} < \infty$ for these.

Let us study $\|v_j\|_{L^1}$ in more detail. Again, assuming $N_j \geq N$, by Taylor's theorem

$$v_j(x) = \sum_{|\alpha|=N} \frac{\partial^\alpha v_j(x_c)}{\alpha!} (x - x_c)^\alpha + R_{v_j,N,x_c}(x).$$

Set $P_{j,N}(x) = \sum_{|\alpha|=N} \partial^\alpha v_j(x_c)x^\alpha/\alpha!$, and so $v_j(x) = P_{j,N}(x - x_c) + R_{v_j,N,x_c}(x)$. Define $\|P_{j,N}\| = \int_{\mathbb{S}^{n-1}} |P_{j,N}(\theta)| d\sigma(\theta)$. Then

$$\|P_{j,N}(\cdot - x_c)\|_{L^1(B)} = \frac{\|P_{j,N}\|}{N + n} r^{N+n}$$

and

$$\begin{aligned} |R_{v_j,N,x_c}(x)| &\leq \sum_{|\beta|=N+1} \frac{|x - x_c|^{N+1}}{\beta!} \max_{|\gamma|=N+1} \max_{|y-x_c|\leq 1} |\partial^\gamma v_j(y)| \\ &\leq C_{N,n} |x - x_c|^{N+1} \max_{|\gamma|=N+1} \max_{|y-x_c|\leq 1} \int_{\mathbb{S}^{n-1}} k^{N+1} |\theta^\gamma| |g_j(\theta)| d\sigma(\theta) \\ &\leq C_{N,k,n} |x - x_c|^{N+1} \|g_j\|_{L^2(\mathbb{S}^{n-1})}. \end{aligned}$$

In other words $\|v_j\|_{L^1(B)} \leq C_{N,k,n,G} (\|P_{j,N}\| + r) r^{N+n}$ if v_j has order $N_j \geq N$ at x_c since we had assumed the uniform bound $\|g_j\|_{L^2(\mathbb{S}^{n-1})} \leq G$. Thus

$$\|v\|_{L^1(B \cap \Omega)} \leq C_\Omega 2^{-j} + C_{N,k,n,G} (\|P_{j,N}\| + r) r^{N+n} \tag{6.2}$$

whenever $N_j \geq N$.

Fix $N = 1$ now. At least one of the following is true: 1) there is a subsequence of v_j for which $N_j \geq 1$, or 2) there is a subsequence for which $N_j = 0$. In the former case we note that $\|P_{j,1}\| \leq C_{n,k,G} < \infty$ by the Herglotz wave formula for v_j , and thus (6.2) implies that v has order 1 at x_c ; a stronger result than in the theorem. So consider case 2) from now on.

We may assume that $N_j = 0$ for all j since we are in case 2). We will use Theorem 3.1. To use (3.1) we need to have normalized incident Herglotz waves, a property which is not necessarily true for v_j . However note that $v_j / \|g_j\|_{L^2(\mathbb{S}^{n-1})}$ is normalized. We have

$$\|v_j\|_{L^2(\Omega)} \geq \|v\|_{L^2(\Omega)} - \|v - v_j\|_{L^2(\Omega)} > 1 - 2^{-j}$$

and

$$\begin{aligned} \|v_j\|_{L^2(\Omega)} &\leq \int_{\mathbb{S}^{n-1}} \|e^{ik\theta \cdot x}\|_{L^2(\Omega,x)} |g_j(\theta)| d\sigma(\theta) \\ &\leq \sqrt{m(\Omega)\sigma(\mathbb{S}^{n-1})} \|g_j\|_{L^2(\mathbb{S}^{n-1})}. \end{aligned}$$

In other words $\|g_j\|_{L^2(\mathbb{S}^{n-1})} \geq 1 / (2\sqrt{m(\Omega)\sigma(\mathbb{S}^{n-1})}) > 0$ when $j \geq 1$. We also know that v_j has order 0 at x_c . Hence by Theorem 3.1

$$\|v_{j,\infty}^s\| \geq \frac{\mathcal{S} \|g_j\|_{L^2(\mathbb{S}^{n-1})}}{\exp \exp c \min(1, \frac{\|P_{j,0}\|}{\|g_j\|_{L^2(\mathbb{S}^{n-1})})^{-\ell}} \geq \frac{\mathcal{S} / (2\sqrt{m(\Omega)\sigma(\mathbb{S}^{n-1})})}{\exp \exp c \min(1, \frac{\|P_{j,0}\|}{G})^{-\ell}}$$

for all j . By (6.1) and the above we see that $\|P_{j,0}\| \rightarrow 0$ as $j \rightarrow \infty$.

By having $N = 0$ in (6.2) and taking the limit $j \rightarrow \infty$ we see that $\|v\|_{L^1(B)} \leq C_{k,n,G} r^{n+1}$. Hence

$$\lim_{r \rightarrow 0} \frac{1}{m(B)} \int_B |v(x)| dx = 0. \quad \square$$

7. Discussion

In this paper, we are concerned with the transmission eigenvalue problem, a type of non elliptic and non self-adjoint eigenvalue problem. We derive intrinsic properties of transmission eigenfunctions by showing that they vanish near corners at the support of the potential function involved. This is proved by an indirect approach, connecting to the wave scattering theory. Indeed, we first show that by using the Herglotz-approximation of a transmission eigenfunction as an incident wave field, the generated scattered wave can have an arbitrarily small energy in its far-field pattern. On the other hand, we establish that with an incident Herglotz wave the scattered far-field pattern has a positive lower bound depending on the Herglotz wave's order of vanishing at a corner. This hints that the transmission eigenfunction should vanish near the corner point. Nevertheless, the rigorous justification of the vanishing property is a highly nontrivial procedure.

To our best knowledge, [Theorem 3.2](#) is the first result in the literature on the intrinsic properties of transmission eigenfunctions. The vanishing behavior obviously carries geometric information of the support of the involved potential function V . Indeed, in inverse scattering theory, an important problem arising in practical application is to infer knowledge of V by measurements of the far-field pattern $u_\infty^s\left(\frac{x}{|x|}; u^i\right)$ (cf. [\[12,22,26,33–35\]](#)). There is relevant study on determining the transmission eigenvalues using knowledge of $u_\infty^s\left(\frac{x}{|x|}; u^i\right)$ (cf. [\[8\]](#)). Clearly, it would be interesting and useful as well to determine the corresponding eigenfunctions from the inverse scattering point of view. Indeed, as suggested by [Theorem 3.2](#), if the unknown function V is supported in a convex polyhedral domain, then one might use the vanishing property of the corresponding transmission eigenfunction to determine the vertices of the polyhedral support of V . As mentioned earlier, in the upcoming numerical paper [\[3\]](#), we shall show that the vanishing order is related to the angle of the corner and the vanishing behavior also occurs at the edge singularities of $\text{supp}(V)$. Hence, one can use these intrinsic properties of transmission eigenfunctions to determine the polyhedral support of an unknown function V . This is beyond the aim and scope of the present article and we shall investigate this interesting issue in our upcoming papers.

We will comment on the requirement of uniformly bounded Herglotz kernels of [Theorem 3.2](#). It is a technical condition and very difficult to relate directly to [Theorem 2.1](#). This study is a first step in the research of intrinsic properties of transmission eigenfunctions and we have brought a new phenomenon into attention. This observation was derived from the apparent contradiction of the well-known [Theorem 2.1](#) and our new [Theorem 3.1](#). In addition, the upcoming numerical study [\[3\]](#) gives evidence that this

vanishing phenomenon is true more generally. Also in another upcoming paper (Proposition 3.5 in [2]) we study corner scattering with more general incident waves, namely waves in H^2 that do not need to be defined outside a small interior neighborhood of a corner of Ω . That result suggests that the condition of approximation by uniformly bounded kernels can be swapped out for the condition that v restricted to $\Omega \cap B(x_c, \varepsilon)$ is in H^2 . In other words, if a transmission eigenfunction is smooth enough near a corner, then it must vanish at that corner. We shall further explore this interesting issue in forthcoming papers.

Finally, we would like to mention that [Theorem 3.1](#) is of significant interest for its own sake, particularly for invisibility cloaking (cf. [18,19]). Indeed, it generalizes our earlier corner scattering result in [1] where the incident wave fields are confined to be plane waves. It suggests that if the support of the underlying scatterer possesses corner singularities, then in principle for any incident fields, invisibility cannot be achieved. On the other hand, it also suggests that if one intends to diminish the scattering effect, then the incident wave field should be such chosen that it vanishes to a high order at the corner point. This is another interesting topic worth of further investigation, especially the corresponding extension to anisotropic scatterers.

Acknowledgments

We are grateful to Professor Fioralba Cakoni for helpful discussion on [Proposition 4.2](#) which inspires this article. The work of H Liu was supported by the startup and FRG funds from Hong Kong Baptist University, the Hong Kong RGC grants, No. 12302415 and No. 12302017.

References

- [1] E. Blåsten, H. Liu, On corners scattering stably, nearly non-scattering interrogating waves, and stable shape determination by a single far-field pattern, arXiv:1611.03647.
- [2] E. Blåsten, H. Liu, Recovering piecewise constant refractive indices by a single far-field pattern, arXiv:1705.00815.
- [3] E. Blåsten, X. Li, H. Liu, Y. Wang, On vanishing and localizing of transmission eigenfunctions near singular points: a numerical study, *Inverse Probl.* 33 (2017) 105001.
- [4] E. Blåsten, L. Päiväranta, Completeness of generalized transmission eigenstates, *Inverse Probl.* 29 (10) (2013) 104002.
- [5] E. Blåsten, L. Päiväranta, J. Sylvester, Corners always scatter, *Comm. Math. Phys.* 331 (2014) 725–753.
- [6] F. Cakoni, 2016, private discussion.
- [7] F. Cakoni, D. Gintides, H. Haddar, The existence of an infinite discrete set of transmission eigenvalues, *SIAM J. Math. Anal.* 42 (2010) 237–255.
- [8] F. Cakoni, H. Haddar, Transmission eigenvalues in inverse scattering theory, in [35], 529–578.
- [9] G. Chen, P. Morris, J. Zhou, Visualization of special eigenmodes shapes of a vibrating elliptical membrane, *SIAM Rev.* 36 (1994) 453–469.
- [10] D. Colton, A. Kirsch, A simple method for solving inverse scattering problems in the resonance region, *Inverse Probl.* 12 (4) (1996) 383–393.
- [11] D. Colton, A. Kirsch, L. Päiväranta, Far-field patterns for acoustic waves in an inhomogeneous medium, *SIAM J. Math. Anal.* 20 (1989) 1472–1482.

- [12] D. Colton, R. Kress, *Inverse Acoustic and Electromagnetic Scattering Theory*, 2nd ed., Springer, New York, 1998.
- [13] D. Colton, P. Monk, The inverse scattering problem for time-harmonic acoustic waves in an inhomogeneous medium, *Quart. J. Mech. Appl. Math.* 41 (1988) 97–125.
- [14] J. Elschner, G. Hu, Corners and edges always scatter, *Inverse Probl.* 31 (2015) 015003.
- [15] J. Elschner, G. Hu, Acoustic scattering from corners, edges and circular cones, arXiv:1603.05186.
- [16] J. Gell-Redman, J. Hassel, Potential scattering and the continuity of phase-shifts, *Math. Res. Lett.* 19 (2012) 719–729.
- [17] D. Grebenkov, B. Ngyuen, Geometrical structure of Laplacian eigenfunctions, *SIAM Rev.* 55 (4) (2013) 601–667.
- [18] A. Greenleaf, Y. Kurylev, M. Lassas, G. Uhlmann, Invisibility and inverse problems, *Bull. Amer. Math. Soc.* 46 (2009) 55–97.
- [19] A. Greenleaf, Y. Kurylev, M. Lassas, G. Uhlmann, Cloaking devices, electromagnetic wormholes and transformation optics, *SIAM Rev.* 51 (2009) 3–33.
- [20] S. Heilman, R. Strichartz, Localized eigenfunctions: here you see them, there you don't, *Notices Amer. Math. Soc.* 5 (2010).
- [21] G. Hu, M. Salo, E. Vesalainen, Shape identification in inverse medium scattering, *SIAM J. Math. Anal.* 48 (2016) 152–165.
- [22] V. Isakov, *Inverse Problems for Partial Differential Equations*, 2nd ed, Springer-Verlag, New York, 2006.
- [23] J. Keller, S. Rubinow, Asymptotic solution of eigenvalue problems, *Ann. Phys.* 9 (1960) 24–75.
- [24] A. Kirsch, N. Grinberg, *The Factorization Method for Inverse Problems*, Oxford Lecture Series in Mathematics and Its Applications, vol. 36, Oxford University Press, Oxford, 2008.
- [25] E. Lakshtanov, B. Vainberg, Applications of elliptic operator theory to the isotropic interior transmission eigenvalue problem, *Inverse Probl.* 29 (10) (2013) 104003.
- [26] A. Nachman, Reconstructions from boundary measurements, *Ann. of Math.* (2) 128 (1988) 531–576.
- [27] S. Nazarov, Localization near the corner point of the principal eigenfunction of the Dirichlet problem in a domain with thin edging, *Sib. Math. J.* 52 (2) (2011) 274–290.
- [28] B. Nguyen, Localization of Laplacian eigenfunctions in simple and irregular domains, *Mathematical Physics [math-ph]*, Ecole Polytechnique X (2012) English. <paste1-00764806>.
- [29] L. Päivärinta, M. Salo, E. Vesalainen, Strictly convex corners scatter, *Rev. Mat. Iberoam.* (2017), in press.
- [30] L. Päivärinta, J. Sylvester, Transmission eigenvalues, *SIAM J. Math. Anal.* 40 (2008) 738–753.
- [31] L. Robbiano, Spectral analysis of the interior transmission eigenvalue problem, *Inverse Probl.* 29 (10) (2013) 104001.
- [32] B. Rynne, B. Sleeman, The interior transmission problem and inverse scattering from inhomogeneous media, *SIAM J. Math. Anal.* 22 (1991) 1755–1762.
- [33] J. Sylvester, G. Uhlmann, A global uniqueness theorem for an inverse boundary value problem, *Ann. of Math.* (2) 125 (1987) 153–169.
- [34] G. Uhlmann, *Visibility and Invisibility*, ICIAM 07—6th International Congress on Industrial and Applied Mathematics, Eur. Math. Soc., Zürich, 2009, pp. 381–408.
- [35] G. Uhlmann (Ed.), *Inverse Problems and Applications: Inside Out II*, MSRI Publications, vol. 60, Cambridge University Press, 2013.
- [36] N. Weck, Approximation by Herglotz wave functions, *Math. Methods Appl. Sci.* 27 (2004) 155–162.