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Well-posedness of the Goursat problem and stability for point source inverse backscattering

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Abstract

We show logarithmic stability for the point source inverse backscattering problem under the assumption of angularly controlled potentials. Radial symmetry implies Hölder stability. Importantly, we also show that the point source equation is well-posed and also that the associated characteristic initial value problem, or Goursat problem, is well-posed. These latter results are difficult to find in the literature in the form required by the stability proof.

Keywords: inverse backscattering, point source, Goursat problem, stability

(Some figures may appear in colour only in the online journal)

1. Introduction

For a potential function q supported inside the unit disc B in $\mathbb{R}^3$ and a point a consider the point source problem

$$(\partial_t^2 - \Delta - q)U^a(x,t) = \delta(x-a,t), \qquad x \in \mathbb{R}^3, t \in \mathbb{R}, \tag{1}$$

$$U^a(x,t) = 0, \qquad x \in \mathbb{R}^3, t < 0. \tag{2}$$

We define the point source backscattering data as the function $(a,t) \mapsto U^a(a,t)$. This paper has two goals: to prove the well-posedness of (1) and (2), and then to solve the inverse problem of determining q from the point source backscattering data $U^a(a,t)$ with $a \in \partial B$ and $t > 0$.

The ordinary inverse problem of backscattering for arbitrary potentials is a major open problem. In it the scattering amplitude $A(\hat{x}, \theta, k)$ is measured for frequencies $k \in \mathbb{R}_+$, incident plane-wave directions $|\theta| = 1$, and measurement direction $\hat{x} = -\theta$. The question is whether such data corresponds to a unique potential q. This question has been solved in the time-domain for an admissible class of potentials in [RU1]. For a more in-depth review of earlier results please refer to [MU].

Traditional backscattering applications include radar, fault detection in fiber optics, Rutherford backscattering and x-ray backscattering (e.g. full-body scanners) among others. What's common to all of these is that the measured object (or fault) is located far away from the wave source. From the point of view of the Rakesh–Uhlmann [RU1, RU2] techniques the classical backscattering problem in the time-domain behaves as the point source problem with source at infinity. This means that the problem (1) and (2) models a situation where the wave source is close to the object under investigation, for example in the order of a few wavelengths. Therefor our results imply that backscattering experiments would give useful information even when the object is close. For example one could imagine using the backscattering of sound, radio or elastic waves to find faults in an object of human scale.

Uniqueness for the inverse backscattering problem related to (1) and (2) was shown by Rakesh and Uhlmann for an admissible class of smooth potentials in [RU2]. We shall show stability for their method. In addition we will show that the direct problem is well-posed in the sense of Hadamard, including all the required norm estimates.

The question of well-posedness of the direct problem would seem well-known to the experts at first sight. However this result is very difficult to find in the literature for non-smooth potentials and with explicit norm estimates. We hope that future research on the topic finds the explicit proof convenient.

The main motivation for this paper is the proof of the following stability theorem. As in [RU1, RU2] it applies to a class of potentials whose differences are *angularly controlled*.

Theorem 1.1. *Let $B = B(\bar{0}, 1) \subset \mathbb{R}^3$ and fix positive a priori parameters $S, \mathcal{M} < \infty$ and $h < 1$. Then there are $\mathfrak{C}, \mathfrak{D} < \infty$ with the following properties:*

Let $q_1, q_2 \in C_c^7(B)$ with norm bounds $\|q_j\|_{C^7} \leqslant \mathcal{M}$. Assume moreover that $\operatorname{supp} q_1$ and $\operatorname{supp} q_2$ are no closer than distance h from ∂B. If $q_1 - q_2$ is angularly controlled with constant S, i.e.

$$\sum_{i<j} \int_{|x|=r} |\Omega_{ij}(q_1 - q_2)(x)|^2 \, d\sigma(x) \leqslant S^2 \int_{|x|=r} |(q_1 - q_2)(x)|^2 \, d\sigma(x) \tag{3}$$

for any $0 < r < 1$ where $\Omega_{ij} = x_i \partial_j - x_j \partial_i$ are the angular derivatives, then we have the following conditional stability estimate

$$\|q_1 - q_2\|_{L^2(\{|x|=r\})} \leqslant e^{\mathfrak{C}/r^4} \|U_1^a - U_2^a\| \tag{4}$$

for any given positive r. Here U_1^a and U_2^a are the unique solutions to the problem (1) and (2) given by theorem 1.2 with $a \in \partial B$, $q = q_1$, $q = q_2$, and

$$\|U_1^a - U_2^a\|^2 = \sup_{0<\tau<1} \int_{|a|=1} \left| \partial_\tau \left(\tau (U_1^a - U_2^a)(a, 2\tau) \right) \right|^2 d\sigma(a)$$

is the backscattering measurement norm that we impose.

A fortiori we get the logarithmic full-domain estimate

$$\|q_1 - q_2\|_{L^2(B)} \leqslant \mathfrak{D} \left(\ln \frac{1}{\|U_1^a - U_2^a\|} \right)^{-1/4} \tag{5}$$

when $\|U_1^a - U_2^a\| < e^{-1}$ *and* $\|q_1 - q_2\|_{L^2(B)} \leqslant \mathfrak{D}\,\|U_1^a - U_2^a\|$ *otherwise.*

If instead of angular control for $q_1 - q_2$ *we assume the stronger condition of radial symmetry, we have*

$$\|q_1 - q_2\|_{L^2(\{|x|=r\})} \leqslant \mathfrak{C} r^\alpha \,\|U_1^a - U_2^a\|$$

where $\alpha = \alpha(\mathcal{M}, h, B)$*, and this implies the full domain Hölder estimate*

$$\|q_1 - q_2\|_{L^2(B)} \leqslant \mathfrak{D}\,\|U_1^a - U_2^a\|^{\frac{1}{1+\alpha}}.$$

The proof of the above theorem is presented in section 4 and is based on the innovative techniques from [RU2]. It starts with writing the data $U_1^a(a, 2\tau) - U_2^a(a, 2\tau)$ as an integral involving $q_1 - q_2$ and solutions to (1) and (2). The linear part of this integral is the average of $q_1 - q_2$ over spheres with centers on ∂B. Proposition 4.2 is key for inverting the linearised problem and its perturbations. The inversion formula to this, and to the corresponding linearized problem in plane-wave inverse backscattering—which is the Radon transform—is an ill-posed operator. Angular control and Grönwall's inequality give uniqueness and logarithmic stability to the linearized problem, and also to the full nonlinear inverse problem.

From the point of view of applications the logarithmic stability seems unpleasant. If we knew in advance that $q_1 = q_2$ in a fixed neighbourhood of the origin, then (4) would give us a Lipschitz stability estimate $\|q_1 - q_2\|_{L^2(B)} \leqslant C\,\|U_1^a - U_2^a\|$. However it is not clear under which conditions $q_1 - q_2$ would stay angularly controlled if the origin was moved to another location, e.g. outside of their supports. The method of this paper and [RU1, RU2] is centered around angular control so further work should focus on understanding this condition. When the integrals that use this condition are ignored, as happens when $q_1 - q_2$ is radially symmetric, we get Hölder stability.

It would be extremely surprising if Hölder stability was possible in general. The fixed frequency multi-static inverse problem is known to be exponentially ill-posed [Man]. Counting dimensions, this problem is overdetermined in $\mathbb{R}^3$ while the harder backscattering problem is determined. However no formal inference can be made since there is no known direct way of deducing the multi-frequency (or time-domain) backscattering data from the fixed frequency multi-static data. Furter comments on this complex issue deserve a completely new study.

Showing the well-posedness of the direct problem (1) and (2) is a major effort. This has to be done for two reasons. Firstly because the proof of theorem 1.1 requires norm-estimates related to the solution U^a. These estimates are lacking from the literature. Secondly, it makes sure that the backscattering data $U^a(a, t)$ is smooth enough for the above theorem to say anything meaningful.

Theorem 1.2. *Let* $n \geqslant 7$ *and* $B = B(\bar{0}, 1)$ *be the unit disc in* $\mathbb{R}^3$*. Let* $q \in C_c^n(B)$ *and* $a \in \partial B$*. Then the point source problem (1) and (2) has a unique solution* U^a *in the set of distributions of order n. It is given by*

$$U^a(x, t) = \frac{\delta(t - |x - a|)}{4\pi\,|x - a|} + H(t - |x - a|) r^a(x, t) \tag{6}$$

where $r^a \in C^1(\mathbb{R}^3 \times \mathbb{R})$ *and* δ, H *are the Dirac-delta distribution and Heaviside function on* $\mathbb{R}$*. For any* $T > 0$ *and* $\mathcal{M} \geqslant \|q\|_{C^7}$ *it has the norm estimate*

$$\|r^a\|_{C^1(\mathbb{R}^3 \times [0,T])} \leqslant C_{T,\mathcal{M}}. \tag{7}$$

Moreover U^a is C^1-smooth outside the light cone $t = |x - a|$. In particular the map $(a, \tau) \mapsto U^a(a, 2\tau)$ is well-defined $\partial B \times (0, 1) \to \mathbb{C}$ and continuously differentiable in τ. Furthermore

$$\sup_{a \in \partial B} \sup_{0<\tau<1} \left| \partial_\tau^\beta (U_1^a - U_2^a)(a, 2\tau) \right| \leqslant C_{\mathcal{M}} \|q_1 - q_2\|_{C^7}$$

for solutions U_j^a arising from two potentials q_j, $j = 1, 2$ and for any $\beta \in \{0, 1\}$.

The proof of the above will be done by a *progressive wave expansion*. This will lead us to a characteristic initial value problem called the *Goursat problem*. In [RU2] this problem was mentioned briefly with reference to [Rom]. Another well-known source on the point source problem is [Fri]. The former studies the point source problem in low regularity Sobolev spaces, which is not good enough since we need a uniform ∂_t-estimate. The latter suffers from too much generality and considers only C^∞ smooth coefficients, without any norm estimates. Neither reference mentions the Goursat problem by name or defines it explicitly.

There are other sources, more focused on the Goursat problem. For example [Cag] is very detailed on the topic but seems to have slightly larger smoothness requirements than we do. See also [Bal1, Bal2] for a very detailed analysis but their model has a region removed from the middle of the characteristic cone. Therefor we shall also prove well-posedness of the Goursat problem.

Theorem 1.3. *For $n \in \mathbb{N}, n \geqslant 5$ let $q \in C^n(\mathbb{R}^3)$ and $g \in C^{n+2}(\mathbb{R}^3)$ with the norm bounds $\|q\|_{C^n} \leqslant \mathcal{M}$ and $\|g\|_{C^{n+2}} \leqslant \mathcal{N}$. Then there is a unique C^1 solution u to the problem*

$$\begin{aligned} (\partial_t^2 - \Delta - q)u &= 0, \qquad x \in \mathbb{R}^3, t > |x| \\ u(x, t) &= g(x), \qquad x \in \mathbb{R}^3, t = |x| \,. \end{aligned}$$

It is also in $C^s(\mathbb{R}^3 \times \mathbb{R})$ where $s = \lfloor \frac{n-2}{3} \rfloor$ and satisfies

$$(\partial_t + \partial_r)u = \partial_r g, \qquad x \in \mathbb{R}^3, t = |x|$$

where $\partial_r = \frac{x}{|x|} \cdot \nabla_x$.

For any $T < \infty$ the solution has the norm estimate

$$\|u\|_{C^s(\mathbb{R}^3 \times [0,T])} \leqslant C_{T,n,\mathcal{M}} \mathcal{N}.$$

Finally, if $q_1, q_2 \in C^n(\mathbb{R}^3)$ and $g_1, g_2 \in C^{n+2}(\mathbb{R}^3)$ then their corresponding solutions satisfy

$$\|u_1 - u_2\|_{C^s(\mathbb{R}^3 \times [0,T])} \leqslant C_{T,n,\mathcal{M},\mathcal{N}} \big(\|q_1 - q_2\|_{C^n(\mathbb{R}^3)} + \|g_1 - g_2\|_{C^{n+2}(\mathbb{R}^3)} \big).$$

We will use the following notation for function spaces of continuous functions.

Definition 1.4. Let $s \in \mathbb{N}$ and $X \subset \mathbb{R}^d$ for some $d \in \mathbb{Z}_+$. The set $C^s(X)$ contains all $f : X \to \mathbb{C}$ that are s times continuously differentiable. A subscript of c as in $C_c^s(X)$ indicates compact support in X.

Given $s, \tau \in \mathbb{N}$ we denote by $C^{s,\tau}(\mathbb{R}^3 \times \mathbb{R})$ the space of continuous functions $f : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{C}$ for which $\partial_x^\alpha \partial_t^\beta f$ is continuous when $\alpha_1 + \alpha_2 + \alpha_3 \leqslant s$ and $\beta \leqslant \tau$.

For estimates,

$$\|f\|_{C^s(X)} = \sum_{|\alpha|\leqslant s} \sup_{p\in X} |\partial^\alpha f(p)|$$

$$\|f\|_{C^{s,\tau}(X)} = \sum_{\substack{|\alpha|\leqslant s\\ \beta\leqslant\tau}} \sup_{(x,t)\in X} \left|\partial_x^\alpha \partial_t^\beta f(x,t)\right|$$

where α is a multi-index of appropriate dimension.

A priori no uniform bounds are required above. The solution to the wave equation has finite speed of propagation so the qualitative statements of our results stay true even for continuous but unbounded functions.

2. Goursat problem

The goal of this section is simple: prove the well-posedness of the Goursat problem, including norm estimates of the solution with dependence on the potential q and Dirichlet data g on the characteristic cone. Before that we will show informally how the point source problem is reduced to the *Goursat problem*, or *characteristic initial-boundary value problem*. Lemma 3.1 validates these informal calculations.

If $\delta, H \in \mathscr{D}'(\mathbb{R})$ are the delta-distribution and Heaviside function, then applying the operator $\partial_t^2 - \Delta + q$ to the ansatz

$$U^a(x,t) = \frac{\delta(t-|x-a|)}{4\pi\,|x-a|} + H(t-|x-a|)r^a(x,t) \tag{8}$$

gives

$$(\partial_t^2 - \Delta - q)U^a = (\partial_t^2 - \Delta)\frac{\delta(t-|x-a|)}{4\pi\,|x-a|} - \frac{q(x)\delta(t-|x-a|)}{4\pi\,|x-a|}$$
$$+\,\delta'(t-|x-a|)(r^a - r^a) + 2\frac{\delta(t-|x-a|)}{|x-a|}\left(|x-a|\,\partial_t r^a + r^a + (x-a)\cdot\nabla r^a\right)$$
$$+\,H(t-|x-a|)(\partial_t^2 - \Delta - q)r^a.$$

Now U^a will be a solution to (1) and (2) if

$$(\partial_t^2 - \Delta - q)r^a = 0, \qquad x\in\mathbb{R}^3, t > |x-a|\,,$$
$$(|x-a|\,\partial_t + 1 + (x-a)\cdot\nabla)\,r^a = \frac{q}{8\pi}, \qquad x\in\mathbb{R}^3, t = |x-a|\,.$$

However if $F(x) = |x-a|\,r^a(x,|x-a|)$ then the chain rule shows that

$$\frac{x-a}{|x-a|}\cdot\nabla F = (|x-a|\,\partial_t + 1 + (x-a)\cdot\nabla)\,r^a(x,|x-a|) = \frac{q(x)}{8\pi} \tag{9}$$

and solving for F gives

$$r^a(x,|x|) = \frac{1}{8\pi}\int_0^1 q(a+s(x-a))\mathrm{d}s. \tag{10}$$

Proving the converse requires more assumptions, so we will skip it now. Instead we shall show that the Goursat problem

$$(\partial_t^2 - \Delta - q)r^a = 0, \qquad x \in \mathbb{R}^3, t > |x - a|, \tag{11}$$

$$r^a = g, \qquad x \in \mathbb{R}^3, t = |x - a| \tag{12}$$

has a unique solution in C^1 for any q and g smooth enough, and that this solution also satisfies the boundary condition (9) when g is chosen from (10). Natural smoothness conditions are $q \in C^n$ and $g \in C^{n+2}$.

Definition 2.1. For $k \in \mathbb{Z}$ define the function $\mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$

$$\gamma^k(x,t) = \begin{cases} \frac{(t^2-|x|^2)^k}{k!}, & k \in \mathbb{N} \\ 0, & k < 0 \end{cases}.$$

Lemma 2.2. *For $n \in \mathbb{N}$ let $q \in C^n(\mathbb{R}^3)$ and $g \in C^{n+2}(\mathbb{R}^3)$. Let $m \leqslant \lfloor \frac{n}{2} \rfloor + 1$ be an integer. Then define $v\colon \mathbb{R}^3 \times \mathbb{R} \to \mathbb{C}$ by*

$$v(x,t) = \sum_{k=0}^{m} a_k(x)\gamma^k(x,t)$$

where the functions a_k are defined as

$$a_0(x) = g(x), \qquad \mathbb{R}^3, \tag{13}$$

$$a_{k+1}(x) = \frac{1}{4}\int_0^1 s^{k+1}\big((q+\Delta)a_k\big)(xs)\mathrm{d}s, \qquad \mathbb{R}^3. \tag{14}$$

Then $a_k \in C^{n+2-2k}(\mathbb{R}^3)$. They have the norm estimate

$$\|a_k\|_{C^{n+2-2k}(\mathbb{R}^3)} \leqslant \left(\frac{1+\|q\|_{C^n(\mathbb{R}^3)}}{4}\right)^k \|g\|_{C^{n+2}(\mathbb{R}^3)}.$$

If $q_1, q_2 \in C^n(\mathbb{R}^3)$ and $g_1, g_2 \in C^{n+2}(\mathbb{R}^3)$ then for the corresponding sequences a_{k1} and a_{k2} we have

$$\|a_{k1} - a_{k2}\|_{C^{n+2-2k}} \leqslant \big(1+\mathcal{M}\big)^k \|g_1 - g_2\|_{C^{n+2}} + k\big(1+\mathcal{M}\big)^{k-1}\mathcal{N}\,\|q_1 - q_2\|_{C^n}$$

whenever $\mathcal{M} \geqslant \|q_j\|_{C^n}$ and $\mathcal{N} \geqslant \|g_j\|_{C^{n+2}}$. Moreover

$$(\partial_t^2 - \Delta - q)v = -(q+\Delta)a_m\gamma^m, \qquad x \in \mathbb{R}^3, t \in \mathbb{R},$$
$$v(x,t) = g(x), \qquad x \in \mathbb{R}^3, t = \pm|x|.$$

Proof. Let us start by showing the norm estimates. Obviously $a_0 \in C^{n+2}(\mathbb{R}^3)$ with estimate $\|a_0\|_{C^{n+2}} = \|g\|_{C^{n+2}}$ and $a_{0j} \to a_0$ in norm. Assume that $a_k \in C^{n+2-2k}$. Then qa_k has smoothness $\min(n, n+2-2k)$, and Δa_k has smoothness $n-2k$. Hence a_{k+1} has smoothness $n-2k$ at worst, with norm estimate

$$\|a_{k+1}\|_{C^{n-2k}} \leqslant \frac{1}{4}(1+\|q\|_{C^n})\,\|a_k\|_{C^{n+2-2k}}$$

whose coefficient could be improved by taking into account the value of the integral $\int_0^1 s^{k+1}\mathrm{d}s$. The norm estimate for a general k is

$$\|a_k\|_{C^{n+2-2k}} \leqslant \left(\frac{1+\|q\|_{C^n}}{4}\right)^k \|g\|_{C^{n+2}}$$

by induction.

For the difference we note that

$$\big(a_{(k+1)1} - a_{(k+1)2}\big)(x) = \frac{1}{4}\int_0^1 s^{k+1}\big((q_1+\Delta)a_{k1} - (q_2+\Delta)a_{k2}\big)(xs)\mathrm{d}s$$

and thus

$$\begin{aligned}&\|a_{(k+1)1} - a_{(k+1)2}\|_{C^{n-2k}}\\ &\quad\leqslant (1+\|q_1\|_C^n)\,\|a_{k1}-a_{k2}\|_{C^{n+2-2k}} + \|q_1-q_2\|_{C^n}\,\|a_{k2}\|_{C^{n-2k}}\\ &\quad\leqslant \big(1+\mathcal{M}\big)\,\|a_{k1}-a_{k2}\|_{C^{n+2-2k}} + \big(1+\mathcal{M}\big)^k\mathcal{N}\,\|q_1-q_2\|_{C^n}\end{aligned}$$

in terms of the *a priori* bounds. The norm estimate for the difference is now a simple induction.

The claim $(\partial_t^2 - \Delta - q)v = -(q+\Delta)a_m\gamma^m$ follows from noting that $a_0 = g$, $4x\cdot\nabla a_{k+1} + 4(2+k)a_{k+1} - (q+\Delta)a_k = 0$, and $\partial_t\gamma^k = 2t\gamma^{k-1}$, $\nabla\gamma^k = -2x\gamma^{k-1}$, and then finally applying $\partial_t^2 - \Delta - q$ to the definition of v. □

Lemma 2.3. *Let $n,\tau\in\mathbb{N}$, $q\in C^n(\mathbb{R}^3)$ and $F\in C^{n,\tau}(\mathbb{R}^3\times\mathbb{R})$. Assume that $F(x,t)=0$ when $t<|x|$, and consider the problem*

$$(\partial_t^2 - \Delta - q)w = F, \qquad x\in\mathbb{R}^3, t\in\mathbb{R}, \tag{15}$$

$$w = 0, \qquad x\in\mathbb{R}^3, t<0. \tag{16}$$

It has a solution $w\in C^{n,\tau}(\mathbb{R}^3\times\mathbb{R})$ which moreover vanishes on $t<|x|$. Given $T<\infty$ and $\mathcal{M}\geqslant\|q\|_{C^n(\mathbb{R}^3)}$ it satisfies

$$\|w\|_{C^{n,\tau}(\mathbb{R}^3\times[0,T])} \leqslant C_{T,n,\mathcal{M}}\,\|F\|_{C^{s,\tau}(\mathbb{R}^3\times[0,T])}$$

where

$$C_{T,n,\mathcal{M}} = C_{n,\tau}\sum_{m=0}^{\infty}\frac{C_n^m\mathcal{M}^mT^{2(m+1)}}{4^{m+1}(m+1)!(m+2)!} < \infty$$

and $C_{n,\tau}$ and C_n are finite and depend only on the parameters in their indices.

Finally, given such q_1,q_2 and F_1,F_2 let w_1,w_2 be the corresponding solutions. With the a priori bounds $\|q_j\|_{C^n(\mathbb{R}^3)}\leqslant\mathcal{M}$ and $\|F_j\|_{C^{n,\tau}(\mathbb{R}^3\times[0,T])}\leqslant\mathcal{N}$ we have

$$\|w_1-w_2\|_{C^{n,\tau}(\mathbb{R}^3\times[0,T])} \leqslant C_{T,n,\mathcal{M},\mathcal{N}}\big(\,\|F_1-F_2\|_{C^{n,\tau}(\mathbb{R}^3\times[0,T])} + \|q_1-q_2\|_{C^n(\mathbb{R}^3)}\,\big)$$

where $C_{T,n,\mathcal{M},\mathcal{N}}$ is finite and depends only on the parameters in its indices.

Proof. Consider the operator

$$Kf(x,t) = \int_{\mathbb{R}^3} \frac{f(x-y, t-|y|)}{4\pi |y|} \mathrm{d}y$$

giving $(\partial_t^2 - \Delta)Kf = f$ for compactly supported distributions $f \in \mathscr{E}'(\mathbb{R}^3 \times \mathbb{R})$ and $Kf(x,t) = 0$ for $t < \inf_t \operatorname{supp} f$. This is also true for f supported on $|x| \leqslant t$ (see theorem 4.1.2 in [Fri]) and then the integration area becomes $|x-y| + |y| \leqslant t$. By lemma 5.4

$$\left|\partial_x^\alpha \partial_t^\beta Kf(x,t)\right| \leqslant \begin{cases} \sup_{\mathbb{R}^3 \times]-\infty, t[} \left|\partial_x^\alpha \partial_t^\beta f\right| \frac{t^2 - |x|^2}{8}, & t > |x| \\ 0, & t \leqslant |x| \end{cases}$$

when $\partial_x^\alpha \partial_t^\beta f$ is a continuous function. In essence Kf has the same smoothness properties as f.

The equation $(\partial_t^2 - \Delta - q)w = F$ with $w = 0$ for negative time is equivalent to $w = KF + K(qw)$. Set $w_0(x,t) = KF(x,t)$ and $w_{m+1} = K(qw_m)$, and we will build the final solutions as

$$w = \sum_{m=0}^{\infty} w_m.$$

We see immediately by the properties of K that $w_m \in C^{n,\tau}(\mathbb{R}^3 \times \mathbb{R})$ for all m and that they vanish on $t < |x|$. Moreover

$$\left|\partial_x^\alpha \partial_t^\beta w_0(x,t)\right| \leqslant \sup_{\mathbb{R}^3 \times [0,t]} \left|\partial_x^\alpha \partial_t^\beta F\right| \frac{t^2 - |x|^2}{8}$$

when $t > |x|$ and $\alpha_1 + \alpha_2 + \alpha_3 \leqslant n$, $\beta \leqslant \tau$.

Let us prove the claim by induction. Assume that for any $\alpha_1 + \alpha_2 + \alpha_3 \leqslant n$ and $\beta \leqslant \tau$ we have

$$\left|\partial_x^\alpha \partial_t^\beta w_m(x,t)\right| \leqslant C_m \|q\|_{C^n(\mathbb{R}^3)}^m \|F\|_{C^{n,\tau}(\mathbb{R}^3 \times [0,t])} (t^2 - |x|^2)^{m+1} \tag{17}$$

for some C_m which might depend on the other parameters. Then recall $w_m = 0$ for $t < |x|$ and the definition of w_{m+1}. We get

$$\begin{aligned} \left|\partial_x^\alpha \partial_t^\beta w_{m+1}(x,t)\right| &= \left|\int_{\mathbb{R}^3} \frac{\partial_x^\alpha \left(q(x-y)\partial_t^\beta w_m(x-y, t-|y|)\right)}{4\pi |y|} \mathrm{d}y\right| \\ &\leqslant C_m \sum_{\gamma \leqslant \alpha} \binom{\alpha}{\gamma} \|q\|_{C^n(\mathbb{R}^3)}^{m+1} \|F\|_{C^{n,\tau}(\mathbb{R}^3 \times [0,t])} \\ &\quad \cdot \int_{|x-y|+|y| \leqslant t} \frac{\left((t-|y|)^2 - |x-y|^2\right)^{m+1}}{4\pi |y|} \mathrm{d}y \\ &= \frac{C_m C_{s,n}}{4(m+2)(m+3)} \|q\|_{C^n(\mathbb{R}^3)}^{m+1} \|F\|_{C^{n,\tau}(\mathbb{R}^3 \times [0,t])} (t^2 - |x|^2)^{m+2} \end{aligned}$$

where the last equality comes from lemma 5.4, and where

$$C_n = \max_{|\alpha|\leqslant n} \sum_{\gamma\leqslant\alpha} \binom{\alpha}{\gamma}.$$

We also have $w_{m+1}(x,t) = 0$ for $t < |x|$. Hence we have the recursion formula $C_{m+1} = C_m C_n/(4(m+2)(m+3))$ and $C_0 = 1/8$. This implies that (17) holds with

$$C_m = \frac{C_n^m}{4^{m+1}(m+1)!(m+2)!}$$

for $m = 0, 1, \ldots$.

The series

$$\sum_{m=0}^{\infty} \left|\partial_x^\alpha \partial_t^\beta w_m(x,t)\right| \leqslant \sum_{m=0}^{\infty} \frac{C_n^m \|q\|_{C^n(\mathbb{R}^3)}^m (t^2-|x|^2)^{m+1}}{4^{m+1}(m+1)!(m+2)!} \|F\|_{C^{n,\tau}(\mathbb{R}^3\times[0,t])}$$

converges uniformly for any t, $|x|$ under a given bound, so the function w is well defined. Note that the extension of $t^2 - |x|^2$ by zero to $t < |x|$ is continuous. Hence $\partial_x^\alpha \partial_t^\beta w$ is continuous in $\mathbb{R}^3 \times \mathbb{R}$ when $\alpha_1 + \alpha_2 + \alpha_3 \leqslant n$ and $\beta \leqslant \tau$. Thus $w \in C^{n,\tau}(\mathbb{R}^3 \times \mathbb{R})$.

The final claim, continuous dependence on q and F, follows from the previous estimates. Namely, we note that w_1 and w_2 satisfy the assumptions of the source term F, and the difference $w_1 - w_2$ solves

$$(\partial_t^2 - \Delta - q_1)(w_1 - w_2) = F_1 - F_2 + (q_1 - q_2)w_2$$

with $w_1 - w_2 = 0$ for $t < |x|$. The $C^{n,\tau}(\mathbb{R}^3 \times [0,T])$-norm of the right-hand side is bounded above by

$$C_{T,n,\mathcal{M}}\left(\|F_1 - F_2\|_{C^{n,\tau}} + \|q_1 - q_2\|_{C^n} C_{T,n,\mathcal{M}} \|F_2\|_{C^{n,\tau}} \right)$$

and the claim follows from the *a priori* bound on F_2. □

Lemma 2.4. *Let $u\colon \mathbb{R}^3 \times \mathbb{R} \to \mathbb{C}$ be a C^1-function satisfying*

$$(\partial_t^2 - \Delta - q)u = 0, \qquad x \in \mathbb{R}^3, t > |x|$$
$$u(x,t) = g(x), \qquad x \in \mathbb{R}^3, t = |x|$$

for some $q \in C^0(\mathbb{R}^3)$ and $g \in C^1(\mathbb{R}^3)$. If $g = 0$ then $u = 0$ in $|x| \leqslant t$.

Proof. Define

$$E(t) = \int_{|x|\leqslant t} (|\partial_t u|^2 + |\nabla u|^2 + |u|^2)\mathrm{d}x.$$

We would like to differentiate E with respect to time, however the lack of continuous second derivatives prevents us from doing that directly. Let φ_ε be a mollifier and $u_\varepsilon = \varphi_\varepsilon * u$. Let $E_\varepsilon(t) = \int_{|x|\leqslant t}(|\partial_t u_\varepsilon|^2 + |\nabla u_\varepsilon|^2 + |u_\varepsilon|^2)\mathrm{d}x$. Then

$$E'_\varepsilon(t) = \int_{|x|=t} \left(|\partial_t u_\varepsilon|^2 + |\nabla u_\varepsilon|^2 + |u_\varepsilon|^2 \right) \mathrm{d}\sigma(x) + 2\Re \int_{|x|\leqslant t} \partial_t u_\varepsilon \cdot \overline{\partial_t^2 u_\varepsilon} \mathrm{d}x$$
$$+ 2\Re \int_{|x|\leqslant t} \nabla \partial_t u_\varepsilon \cdot \overline{\nabla u_\varepsilon} \mathrm{d}x + 2\Re \int_{|x|\leqslant t} \partial_t u_\varepsilon \overline{u_\varepsilon} \mathrm{d}x.$$

Integration by parts shows that the third term is equal to

$$2\Re \int_{|x|=t} \frac{x}{|x|} \partial_t u_\varepsilon \cdot \overline{\nabla u_\varepsilon} \mathrm{d}\sigma(x) - 2\Re \int_{|x|\leqslant t} \partial_t u_\varepsilon \overline{\Delta u_\varepsilon} \mathrm{d}x.$$

By combining both equations above and using $\partial_t^2 u_\varepsilon - \Delta u_\varepsilon = \varphi_\varepsilon * (qu)$ we get

$$E'_\varepsilon(t) = \int_{|x|=t} \left(\left| \frac{x}{|x|} \partial_t u_\varepsilon + \nabla u_\varepsilon \right|^2 + |u_\varepsilon|^2 \right) \mathrm{d}\sigma(x)$$
$$+ 2\Re \int_{|x|\leqslant t} \partial_t u_\varepsilon \overline{(u_\varepsilon + \varphi_\varepsilon * (qu))} \mathrm{d}x.$$

Integrate this with respect to time. Since $u_\varepsilon \to u$ in C^1 locally as $\varepsilon \to 0$, we get

$$E(t) = \int_0^t \int_{|x|=s} \left(\left| \frac{x}{|x|} \partial_s u + \nabla u \right|^2 + |u|^2 \right) \mathrm{d}\sigma(x) \mathrm{d}s$$
$$+ \int_0^t 2\Re \int_{|x|\leqslant s} (1 + \overline{q}) \partial_s u \overline{u} \mathrm{d}x \mathrm{d}s.$$

Let us deal with the boundary integral next. Define $u_b(x) = u(x, |x|)$. Then calculus shows that $\nabla u_b(x) = (\nabla u + \frac{x}{|x|} \partial_t u)(x, |x|)$ because $\nabla |x| = x/|x|$. On the other hand the boundary condition of u shows that $u_b = g$. Thus the formula inside the parenthesis above is equal to $|\nabla g|^2 + |g|^2$.

Note that $\int_0^t \int_{|x|=s} f(x) \mathrm{d}x \mathrm{d}s = \int_{|x|\leqslant t} f(x) \mathrm{d}x$ for time-independent functions f. Then, since $2\Re(A\overline{B}) \leqslant |A|^2 + |B|^2$, we get

$$E(t) \leqslant \int_{|x|\leqslant t} \left(|\nabla g|^2 + |g|^2 \right) \mathrm{d}x + (1 + \|q\|_\infty) \int_0^t \int_{|x|\leqslant s} \left(|\partial_s u|^2 + |u|^2 \right) \mathrm{d}x \mathrm{d}s.$$

The last integral has the upper bound $\int_0^t E(s) \mathrm{d}s$. Grönwall's inequality, for example appendix B.2.k in [Evans], shows that $E(t) = 0$ when $g = 0$. □

We are now ready to prove the well-posedness of the Goursat problem in the sense of Hadamard. Strictly speaking the same proof shows existence in C^0 when $q \in C^2$, $g \in C^4$, but then we cannot guarantee uniqueness or the boundary identity that's stated with ∂_t and ∂_r.

Proof of theorem 1.3. This is a consequence of the uniqueness of lemma 2.4, the progressive wave expansion of lemma 2.2 and the initial value problem of lemma 2.3. Let $m = \lfloor (n+1)/3 \rfloor$, which has $m \geqslant 2$ and $n \geqslant 2m + 1$, and set

$$v(x, t) = g(x) + a_1(x)(t^2 - |x|^2) + \cdot + a_m(x)\gamma^m(x, t) \qquad (18)$$

for $(x,t) \in \mathbb{R}^3 \times \mathbb{R}$, as in lemma 2.2. We have $\|a_k\|_{C^{n+2-2k}} \leqslant C_n(1+\mathcal{M})^k\mathcal{N}$ in $\mathbb{R}^3$. Then $v(x,|x|) = g(x)$ but $(\partial_t^2 - \Delta - q)v = -(q+\Delta)a_m\gamma^m$.

Next let

$$F(x,t) = \begin{cases} (q+\Delta)a_m(x)\gamma^m(x,t), & t > |x| \\ 0, & t \leqslant |x| \end{cases} \tag{19}$$

be our source term for an initial value problem. We have $(q+\Delta)a_m \in C^{n-2m}(\mathbb{R}^3)$, but $\chi_{\{t>|x|\}}\gamma^m$ is in $C^{m-1}(\mathbb{R}^3 \times \mathbb{R})$. Hence $F \in C^{n_0,\tau_0}(\mathbb{R}^3 \times \mathbb{R})$ using the notation of lemma 2.3 whenever $n_0 + \tau_0 \leqslant m - 1$ and $n_0 \leqslant \min(n-2m, m-1) = m-1$. In other words when $n_0 + \tau_0 \leqslant s$. Given $T > 0$ the source has the estimate

$$\|F\|_{C^{n_0,\tau_0}(\mathbb{R}^3\times[0,T])} \leqslant C_{T,n,\mathcal{M}}\mathcal{N}.$$

We can also write out the estimate for v now that the smoothness indices are fixed. Note that γ^k is infinitely smooth in $\mathbb{R}^3 \times \mathbb{R}$, and a_m has the worst smoothness among all the coefficient functions in (18). Thus

$$\|v\|_{C^{n_0,\tau_0}(\mathbb{R}^3\times[0,T])} \leqslant C_{T,n,\mathcal{M}}\mathcal{N} \tag{20}$$

too since $n_0 \leqslant m$ and a_k is independent of t.

Let w solve $(\partial_t^2 - \Delta - q)w = F$ in $\mathbb{R}^3 \times \mathbb{R}$ with $w = 0$ for $t < 0$. Lemma 2.3 shows that such a w exists in $C^{n_0,\tau_0}(\mathbb{R}^3 \times \mathbb{R})$ and it has support on $t \geqslant |x|$. Given $T > 0$ it has the norm estimate

$$\|w\|_{C^{n_0,\tau_0}(\mathbb{R}^3\times[0,T])} \leqslant C_{T,n,\mathcal{M}}\mathcal{N} \tag{21}$$

by the estimate on F.

Since $s \geqslant 1$ then $F \in C^{0,1} \cap C^{1,0}$ with support in $t \geqslant |x|$. This implies that $\partial_t w$ and $\nabla_x w$ are continuous. Since $w = 0$ when $t < |x|$ we see that $(\partial_t + \frac{x}{|x|}\cdot\nabla_x)w = 0$ for $t \leqslant |x|$. Next consider v. We see that on $t = |x|$

$$\partial_t\gamma^k(x,t) = \begin{cases} 2t, & k = 1, \\ 0, & k \neq 1 \end{cases}$$

and

$$\nabla_x\gamma^k(x,t) = \begin{cases} -2x, & k = 1, \\ 0, & k \neq 1 \end{cases},$$

so $\partial_t v = 2ta_1$ and $\nabla_x v = \nabla g - 2xa_1(x)$ if $t = |x|$. This implies that

$$\left(\partial_t + \frac{x}{|x|}\cdot\nabla_x\right)v = \frac{x}{|x|}\cdot\nabla g(x)$$

on $t = |x|$.

If we set $u = v + w$, then we see that $u(x,|x|) = g(x)$ and $(\partial_t + \partial_r)u = \partial_r g$ on $t = r = |x|$ because w is continuous in $\mathbb{R}^3 \times \mathbb{R}$ and supported on $t \geqslant |x|$. Moreover $u \in C^s$ since

$$\|u\|_{C^s(\mathbb{R}^3\times[0,T])} \leqslant C \sup_{n_0+\tau_0\leqslant s} \|u\|_{C^{n_0,\tau_0}(\mathbb{R}^3\times[0,T])}$$

and this gives us the required norm estimate from (20) and (21). Finally $(\partial_t^2 - \Delta - q)u = (\partial_t^2 - \Delta - q)v + F = 0$ on $t > |x|$.

The estimate for the difference of solutions $u_1 - u_2$ to two Goursat problems follows from the corresponding estimate for $v_1 - v_2$ of lemma 2.2 and for $w_1 - w_2$ of lemma 2.3. After using the latter note that

$$\|F_1 - F_2\|_{C^{n_0,\tau_0}} \leqslant C_{T,n}\big(1 + \mathcal{M}\big)\,\|a_{m1} - a_{m2}\|_{C^{n_0}} + \|q_1 - q_2\|\,\|a_{m2}\|_{C^{n_0}}$$

holds and thus can be estimated above by the norms of $q_1 - q_2$ and $g_1 - g_2$. □

3. Well-posedness of the point source backscattering measurements

Now that the Goursat problem has been taken care of we can focus on the point source problem. We will show that given a $C_c^7(B)$ potential q there is a unique solution to (1) and (2), and we can define the associated backscattering measurements. Moreover these measurements depend continuously on the potential, with linear modulus of continuity.

Lemma 3.1. *Let $q \in C_c^0(B)$ and $a \in \partial B$. Let $r^a \in C^1(\mathbb{R}^3 \times \mathbb{R})$ solve the problem*

$$(\partial_t^2 - \Delta - q)r^a = 0, \qquad x \in \mathbb{R}^3, t > |x-a|\,,$$

$$(|x-a|\,\partial_t + 1 + (x-a)\cdot\nabla)\,r^a = \frac{q}{8\pi}, \qquad x \in \mathbb{R}^3, t = |x-a|\,.$$

Define

$$U^a(x,t) = \frac{\delta(t-|x-a|)}{4\pi\,|x-a|} + H(t-|x-a|)r^a(x,t)$$

where $\delta, H \in \mathscr{D}'(\mathbb{R})$ are the delta-distribution and Heaviside function. Then U^a is a solution to the point source problem (1) and (2).

Proof. Take the above form of U^a as an ansatz and note that the first term is the Green's function for $\partial_t^2 - \Delta$

$$(\partial_t^2 - \Delta)\frac{\delta(t-|x-a|)}{4\pi\,|x-a|} = \delta(x-a,t) \tag{22}$$

by for example theorem 4.1.1 in [Fri].

Since the function r^a in our ansatz is *a priori* only C^1, we will use a smoothened delta-distribution and Heaviside function. For $\varepsilon > 0$ let $\delta_\varepsilon : \mathbb{R} \to \mathbb{R}$ be smooth, supported in $]0, 2\varepsilon[$, positive, and $\int \delta_\varepsilon = 1$. Let $H_\varepsilon(t) = \int_{-\infty}^t \delta_\varepsilon(s)\mathrm{d}s$. Then δ_ε converges to the delta-distribution as $\varepsilon \to 0$ and H_ε to the Heaviside function. Let our new ansatz be

$$U_\varepsilon(x,t) = \frac{\delta_\varepsilon(t-|x-a|)}{4\pi\,|x-a|} + H_\varepsilon(t-|x-a|)r^a(x,t).$$

Let us calculate the derivatives of the second term in the ansatz next. Note that $\nabla \cdot (x/|x|) = 2/|x|$ in 3D, and so setting $R = H_\varepsilon(t - |x-a|)r^a(t,x)$ we have

$$
\begin{aligned}
\partial_t R &= \delta_\varepsilon(t-|x-a|)r^a + H_\varepsilon(t-|x-a|)\partial_t r^a\\
\partial_t^2 R &= \delta'_\varepsilon(t-|x-a|)r^a + 2\delta_\varepsilon(t-|x-a|)\partial_t r^a + H_\varepsilon(t-|x-a|)\partial_t^2 r^a,\\
\nabla R &= \delta_\varepsilon(t-|x-a|)\left(-\frac{x-a}{|x-a|}\right) r^a + H_\varepsilon(t-|x-a|)\nabla r^a\\
\Delta R &= \delta'_\varepsilon(t-|x-a|)r^a - \delta_\varepsilon(t-|x-a|)\frac{2r^a}{|x-a|}\\
&\quad - \delta_\varepsilon(t-|x-a|)2\frac{x-a}{|x-a|}\cdot\nabla r^a + H_\varepsilon(t-|x-a|)\delta_\varepsilon r^a,\\
qR &= H_\varepsilon(t-|x-a|)qr^a.
\end{aligned}
$$

Take all terms into account next. Then

$$
\begin{aligned}
&(\partial_t^2 - \Delta - q)U_\varepsilon = (\partial_t^2 - \Delta)\frac{\delta_\varepsilon(t-|x-a|)}{4\pi|x-a|} - \frac{q(x)\delta_\varepsilon(t-|x-a|)}{4\pi|x-a|}\\
&+ \delta'_\varepsilon(t-|x-a|)(r^a - r^a) + 2\frac{\delta_\varepsilon(t-|x-a|)}{|x-a|}\left(|x-a|\,\partial_t r^a + r^a + (x-a)\cdot\nabla r^a\right)\\
&+ H_\varepsilon(t-|x-a|)(\partial_t^2 - \Delta - q)r^a.
\end{aligned}
$$

As $\varepsilon \to 0$ the first term above converges to $\delta(x-a,t)$ in the space of distributions. The terms with coefficients δ'_ε and H_ε vanish. The former trivially, and the latter because our choice of δ_ε makes sure that supp $H_\varepsilon \subset \mathbb{R}_+$. In other words

$$
\begin{aligned}
&\lim_{\varepsilon\to 0}(\partial_t^2 - \Delta - q)U_\varepsilon - \delta(x-a,t)\\
&\qquad = \lim_{\varepsilon\to 0} 2\frac{\delta_\varepsilon(t-|x-a|)}{|x-a|}\left(|x-a|\,\partial_t r^a + r^a + (x-a)\cdot\nabla r^a - \frac{q(x)}{8\pi}\right)
\end{aligned}
$$

in $\mathscr{D}'(\mathbb{R}^3 \times \mathbb{R})$.

Denote by $f(x,t)$ the continuous function in parenthesis above. Let $\varphi \in C_c^\infty(\mathbb{R}^3\times\mathbb{R})$ be a test function. Then in the support of φ for every $\mu > 0$ there is $\delta > 0$ such that $|f(x,t)| < \mu$ if $|t - |x-a|| < \delta$. Let $2\varepsilon < \delta$. Then

$$
\left|\int_{\mathbb{R}^3\times\mathbb{R}} \frac{\delta_\varepsilon(t-|x-a|)}{|x-a|} f(x,t)\varphi(x,t)\mathrm{d}x\mathrm{d}t\right| \leqslant \mu\,\|\varphi\|_\infty \int_{\mathrm{supp}\ \varphi} \frac{\delta_\varepsilon(t-|x-a|)}{|x-a|}\mathrm{d}x\mathrm{d}t
$$

and by integrating the t-variable first we get the upper bound

$$
\ldots \leqslant \mu\,\|\varphi\|_\infty \int_{B(a,R_\varphi)} \frac{\mathrm{d}x}{|x-a|} = C_\varphi \mu.
$$

In other words the remaining term in the expansion for $(\partial_t^2 - \Delta - q)U_\varepsilon$ tends to zero in the distribution sense. Hence

$$
(\partial_t^2 - \Delta - q)U_\varepsilon \to \delta(x-a,t)
$$

in $\mathscr{D}'(\mathbb{R}^3 \times \mathbb{R})$. Also, since supp $\delta_\varepsilon \subset \mathbb{R}_+$, it also satisfies the initial condition $U_\varepsilon = 0$ for $t < 0$. Finally, it is easy to see that $U_\varepsilon \to U^a$. Hence the latter is a solution to (1) and (2). □

Lemma 3.2. *For $n \in \mathbb{N}$ let $q \in C^n(\mathbb{R}^3)$ and let U be a distribution of order n on $\mathbb{R}^3 \times \mathbb{R}$ such that $U = 0$ on $t < 0$. If $(\partial_t^2 - \Delta - q)U = 0$ then $U = 0$.*

Proof. Let $\varphi \in C_c^\infty(\mathbb{R}^3 \times \mathbb{R})$ be arbitrary. There is $x_0 \in \mathbb{R}^3$ and $t_0 \in \mathbb{R}$ such that $\varphi(x, t) = 0$ in $|x - x_0| > t_0 - t$, i.e. outside a past light cone. Write $y = x - x_0$ and $s = t_0 - t$, and define

$$Q(y) = q(y + x_0), \qquad F(y, s) = \varphi(y + x_0, t_0 - s).$$

Then $Q \in C^n(\mathbb{R}^3)$, $F \in C_c^\infty(\mathbb{R}^3 \times \mathbb{R})$ and $F(y, s) = 0$ when $s < |y|$. Lemma 2.3 gives the existence of $w \in C^n(\mathbb{R}^3 \times \mathbb{R})$ which vanishes on $s < |y|$ and satisfies $(\partial_s^2 - \Delta - Q)w = F$.

Let

$$\psi(x, t) = w(x - x_0, t_0 - t).$$

Then $\psi(x, t) = 0$ if $|x - x_0| > t_0 - t$. Since $U = 0$ for $t < 0$, the intersection of the supports of ψ and U is a compact set. Since U is of order n and ψ is in C^n their distribution pairing $\langle U, \psi\rangle$ is well defined. Now

$$\begin{aligned}\langle(\partial_t^2 - \Delta - q)U, \psi\rangle &= \langle U, (\partial_t^2 - \Delta - q)\psi\rangle \\ &= \langle\tilde{U}, (\partial_s^2 - \Delta - Q)w\rangle = \langle\tilde{U}, F\rangle = \langle U, \varphi\rangle\end{aligned}$$

where $\tilde{U}$ is the distribution U in the (y, s)-coordinates. Since U is in the kernel of the differential operator and φ is an arbitrary test function, we have $U = 0$. □

Proof of theorem 1.2. Uniqueness follows directly from lemma 3.2. We shall build a solution r^a to the Goursat-type problem of lemma 3.1. We switch boundary conditions as was done at the beginning of section 2. Define

$$g(x) = \frac{1}{8\pi}\int_0^1 q\big(a + s(x - a)\big)\mathrm{d}s$$

and note that $q \in C^n(\mathbb{R}^3)$, $g \in C^{n+2}(\mathbb{R}^3)$ for $n = 5$. The well-posedness of the Goursat problem (theorem 1.3) gives a unique C^1 solution to

$$\begin{aligned}(\partial_t^2 - \Delta - q)r^a &= 0, \qquad x \in \mathbb{R}^3, t > |x - a|, \\ r^a &= g, \qquad x \in \mathbb{R}^3, t = |x - a|.\end{aligned}$$

It has the required norm estimate for any $T > 0$ and in addition it satisfies

$$(\partial_t + \partial_r)r^a = \partial_r g$$

on $t = |x - a|$. Here $r = |x - a|$ and furthermore we denote $\theta = (x - a)/|x - a|$. If in the definition of g we switch integration variables to $s' = rs$ then

$$\partial_r g = -\frac{1}{r}g + \frac{q}{8\pi r}$$

which is well-defined because $q = 0$ in a neighbourhood of a. Recalling that $\partial_r = \theta \cdot \nabla_x$ we see that in fact

$$(|x-a|\,\partial_t + 1 + (x-a)\cdot\nabla_x)r^a = \frac{q}{8\pi}$$

on the boundary $t = |x-a|$. Hence lemma 3.1 shows that U^a is a solution to the point source problem.

The unperturbed Green's function is supported only on $t = |x-a|$. On $t < |x-a|$ the solution vanishes. On $t > |x-a|$ it is equal to r^a which is C^1. In this topology, it depends continuously on a because the Goursat problem depends continuously on the potential and characteristic boundary data. Hence $U(a,2\tau)$ is well-defined for $\tau > 0$ and continuously differentiable in τ.

Let two potentials q_1 and q_2 and their associated solutions r_1^a, r_2^a to the Goursat problem be given. For any $a \in \partial B$ and $\beta \in \{0,1\}$ theorem 1.3 shows the norm estimate

$$\sup_{x\in\mathbb{R}^3}\sup_{0<\tau<1}\left|\partial_\tau^\beta(r_1^a - r_2^a)(x,2\tau)\right| \leqslant C_{\mathcal{M}}\|q_1-q_2\|_{C^7}$$

because $\|g_1-g_2\|_{C^7(\mathbb{R}^3)} \leqslant \|q_1-q_2\|_{C^7(\mathbb{R}^3)}$ and the norms involved are invariant under translations. Letting $x = a$ and then taking the supremum over a proves the claim because $U_1^a - U_2^a = r_1^a - r_2^a$ at $(x,t) = (a,2\tau)$. □

4. Stability of the inverse problem

Now that the direct problem has been shown to be well-defined, including the estimates for the point source backscattering measurements, we can consider the inverse problem. The first step is to write a boundary identity. The following is proven in [RU2] for C^∞-smooth potentials, but it works verbatim in our case too.

Proposition 4.1. *Let $B = B(\bar{0},1)$ be the unit ball in $\mathbb{R}^3$ and $q_1, q_2 \in C_c^7(B)$. Let $a \in \partial B$ and let U_1^a and U_2^a be given by theorem 1.2 for $q = q_j$, $j = 1,2$. Then*

$$U_1^a(a,2\tau) - U_2^a(a,2\tau) = \frac{1}{32\pi^2\tau^2}\int_{|x-a|=\tau}(q_1-q_2)(x)\mathrm{d}\sigma(x) + \int_{|x-a|\leqslant\tau}(q_1-q_2)(x)k(x,\tau,a)\mathrm{d}x \qquad (23)$$

with

$$k(x,\tau,a) = \frac{(r_1^a + r_2^a)(x,2\tau-|x-a|)}{4\pi\,|x-a|} + \int_{|x-a|}^{2\tau-|x-a|} r_1^a(x,2\tau-t)r_2^a(x,t)\mathrm{d}t$$

if $|x-a| \leqslant \tau$.

If we have moreover $\|q_j\|_{C^7} \leqslant \mathcal{M} < \infty$ then

$$\sup_{h\leqslant\tau\leqslant1}\sup_{|a|=1}\int_{|x-a|=\tau}|k(x,\tau,a)|^2\,\mathrm{d}\sigma(x) \leqslant C_{\mathcal{M},h,B} < \infty, \qquad (24)$$

$$\sup_{h\leqslant\tau\leqslant1}\sup_{|a|=1}\int_{h\leqslant|x-a|\leqslant\tau}|\partial_\tau(\tau k(x,\tau,a))|^2\,\mathrm{d}\sigma(x) \leqslant C_{\mathcal{M},h,B} < \infty \qquad (25)$$

for any $h > 0$. Note that $k(x,\tau,a)$ is singular at $x = a$.

Proof. We shall skip the proof of the identities as they have been proved in section 3.2 of [RU2]. It is a matter of calculating

$$\int_{-\infty}^{\infty}\int_{\mathbb{R}^n}(q_1-q_2)(x)U_2^a(x,t)U_1^a(x,2\tau-t)\mathrm{d}x\mathrm{d}t$$

on one hand by integrating by parts, and on the other hand by using the expansion (6). The estimates for k follow directly from (7). □

Our next step is an integral identity related to the first term in (23). The proof for the estimate for $E(a,\tau)$ can be dug from the proofs in [RU2]. We prove it again here, both for clarity, since this estimate might be of interest on its own, and for having an explicit form for the constant in front of the sum.

Proposition 4.2. *Let $Q\in C_c^1(B)$ with B the unit disc in $\mathbb{R}^3$. Then for all $a\in\partial B$ and $0<\tau<|a|$ we have*

$$\partial_\tau\left(\frac{\tau}{4\pi\tau^2}\int_{|x-a|=\tau}Q(s)\mathrm{d}\sigma(x)\right)=\frac{1-\tau}{2}Q\big((1-\tau)a\big)+E(a,\tau) \tag{26}$$

where

$$|E(a,\tau)|^2\leqslant\frac{3}{\pi(1-\tau)}\sum_{i<j}\int_{|x-a|=\tau}\frac{|\Omega_{ij}Q(x)|^2}{\sqrt{|x|-(1-\tau)}}\mathrm{d}\sigma(x).$$

Here the Ω_{ij} are the angular derivatives $x_i\partial_j-x_j\partial_i$ depicted as vector fields in figure 1.

Proof. We may prove the proposition for $Q\in C_c^\infty(B)$ and then get the claim by approximating. Test functions are dense in $C_c^1(B)$ and $\sup|f|+\sup|\nabla f|\leqslant C\,\|f\|_{C^1}$. By proposition 2.1 in [RU2]

$$\partial_\tau\left(\frac{\tau}{4\pi\tau^2}\int_{|x-a|=\tau}Q(s)\mathrm{d}\sigma(x)\right)=\frac{1-\tau}{2}Q\big((1-\tau)a\big)+\frac{1}{4\pi}\int_{|x-a|=\tau}\frac{\alpha\cdot\nabla Q(x)}{\sin\phi}\mathrm{d}\sigma(x),$$

where $\alpha=\alpha(a,x)$ is a unit vector orthogonal to x and ϕ is the angle at the origin between x and a.

Let $T_{ij}=x_ie_j-x_je_i$ so $\Omega_{ij}=T_{ij}\cdot\nabla$. Then for any vector v we have

$$v=\sum_{i<j}\left(v\cdot\frac{T_{ij}}{|x|}\right)\frac{T_{ij}}{|x|}+\left(v\cdot\frac{x}{|x|}\right)\frac{x}{|x|}.$$

On $|x-a|=\tau$ set $v:=\alpha$ and then take the dot product with $\nabla Q(x)$. We get

$$|x|^2\,\alpha\cdot\nabla Q(x)=\sum_{i<j}(\alpha\cdot T_{ij})(T_{ij}\cdot\nabla Q)(x)=\sum_{i<j}(\alpha\cdot T_{ij})\Omega_{ij}Q(x)$$

since $x\perp\alpha$. By the Cauchy–Schwarz inequality

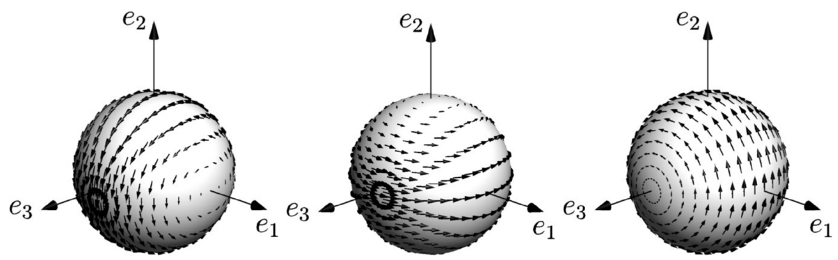

Figure 1. Angular derivatives Ω_{ij}.

$$|\alpha \cdot \nabla Q(x)| \leqslant \frac{|a|}{|x|} \sum_{i<j} |\Omega_{ij} Q(x)|$$

since $|T_{ij}| \leqslant |x|$. This implies

$$|E(a,\tau)| \leqslant \frac{|a|}{4\pi} \sum_{i<j} \int_{|x-a|=\tau} \frac{|\Omega_{ij} Q(x)|}{|x|\,|\sin\phi|} \mathrm{d}\sigma(x).$$

The law of cosines gives us $2\,|a|\,|x|\cos\phi = |a|^2 + |x|^2 - \tau^2$. Solve for $\cos\phi$ to get $\sin\phi = \pm\sqrt{1-\cos^2\phi}$ and hence

$$\frac{1}{|\sin\phi|} = \frac{2\,|a|\,|x|}{\sqrt{4\,|a|^2\,|x|^2 - (|a|^2 + |x|^2 - \tau^2)^2}}$$
$$= \frac{2\,|a|\,|x|}{\sqrt{(|x| - \tau + |a|)(|x| + \tau - |a|)(\tau + |a| - |x|)(\tau + |a| + |x|)}}.$$

But note that by assumption $|a| > \tau > 0$ and $|a| > |x|$ for all $x \in B$. Hence

$$\frac{1}{|\sin\phi|} \leqslant \frac{2\,|a|\,|x|}{\sqrt{|a| - \tau}\sqrt{|x| - (|a| - \tau)}\sqrt{\tau}\sqrt{|a|}}.$$

and we can continue with

$$|E(a,\tau)| \leqslant \frac{|a|^2}{2\pi\sqrt{\tau\,|a|}\sqrt{|a| - \tau}} \sum_{i<j} \int_{|x-a|=\tau} \frac{|\Omega_{ij} Q(x)|}{\sqrt{|x| - (|a| - \tau)}} \mathrm{d}\sigma(x).$$

Finally, use the Cauchy–Schwarz inequality twice: once for $(\sum_{i<j} f_{ij})^2 \leqslant 3\sum_{i<j} f_{ij}^2$ and a second time for the product of the two function $|\Omega_{ij} Q(x)|\,/(|x| - (|a| - \tau))^{1/4}$ and $(|x| - (|a| - \tau))^{-1/4}$. It gives

$$|E(a,\tau)|^2 \leqslant \frac{3\,|a|^3\, I(a,\tau)}{4\pi^2\tau(|a| - \tau)} \sum_{i<j} \int_{|x-a|=\tau} \frac{|\Omega_{ij} Q(x)|^2}{\sqrt{|x| - (|a| - \tau)}} \mathrm{d}\sigma(x)$$

where $I(a,\tau) = \int_{|x-a|=\tau, |x|\leqslant|a|} \mathrm{d}\sigma(x)/\sqrt{|x| - (|a| - \tau)}$.

Parametrize the sphere $|a - x| = \tau$ by $\rho = |x|$ and the azimuth $\theta \in [0, 2\pi]$ to calculate

$I(a, \tau)$. The latter variable gives the inclination of the plane aOx with respect to a fixed reference plane passing through O and a. See figure 2. We also introduce the polar angle ξ. Using the standard spherical coordinates ξ, θ we have

$$\mathrm{d}\sigma(x) = \tau^2 \sin \xi \mathrm{d}\xi \mathrm{d}\theta = \tau^2 \sin \xi \frac{\mathrm{d}\xi}{\mathrm{d}\rho} \mathrm{d}\rho \mathrm{d}\theta.$$

By the law of cosines $|a|^2 + \tau^2 - 2|a| \tau \cos \xi = \rho^2$. Solve for $\cos \xi$ and differentiate this with respect to the variable ρ. Note that a, τ are constants, but $\xi = \xi(\rho)$. We get

$$-\sin \xi \frac{\mathrm{d}\xi}{\mathrm{d}\rho} = \frac{\mathrm{d}}{\mathrm{d}\rho} \cos \xi = -\frac{\rho}{|a| \tau}$$

which implies that $\mathrm{d}\sigma(x) = \tau |a|^{-1} \rho \mathrm{d}\rho \mathrm{d}\theta$.

Thus, since Q vanishes outside B, we have

$$I(a, \tau) = \int_0^{2\pi} \int_{|a|-\tau}^{|a|} \frac{\tau |a|^{-1} \rho \mathrm{d}\rho \mathrm{d}\theta}{\sqrt{\rho - (|a| - \tau)}} \leqslant 2\pi\tau \int_0^{\tau} \frac{\mathrm{d}\rho}{\sqrt{\rho}} = 4\pi\tau^{3/2} \leqslant 4\pi\tau\sqrt{|a|}.$$

Finally use the fact that B is the unit ball and thus $|a| = 1$ to conclude the claim. □

We are now ready to prove stability for point source backscattering.

Proof of theorem 1.1. Write $\tilde{U}^a = U_1^a - U_2^a$ and $\tilde{q} = q_1 - q_2$. By the assumptions and proposition 4.1 we have

$$\tau \tilde{U}^a(a, 2\tau) = \frac{\tau}{32\pi^2\tau^2} \int_{|x-a|=\tau} \tilde{q}(x) \mathrm{d}\sigma(x) + \int_{|x-a|\leqslant\tau} \tilde{q}(x) \tau k(x, \tau, a) \mathrm{d}x$$

for any $\tau > 0$, in particular for $h < \tau < 1$ which we shall assume now. By proposition 4.2 and the differentiation formula for moving regions (e.g. [Evans] appendix C.4) we get

$$\begin{aligned}\partial_\tau \left(\tau \tilde{U}^a(a, 2\tau)\right) &= \frac{1-\tau}{8} \tilde{q}\big((1-\tau)a\big) + \frac{1}{4} E(a, \tau) \\ &+ \int_{|x-a|=\tau} \tilde{q}(x) \tau k(x, t, a) \mathrm{d}\sigma(x) + \int_{|x-a|\leqslant\tau} \tilde{q}(x) \partial_\tau(\tau k(x, \tau, a)) \mathrm{d}x.\end{aligned}$$

By the Cauchy–Schwarz inequalities of $\mathbb{R}^4$ and the L^2-based function spaces $L^2(\{|x-a| = \tau\})$ and $L^2(\{|x-a| \leqslant \tau\})$ we have

$$\begin{aligned}(1-\tau)^2 \left|\tilde{q}\big((1-\tau)a\big)\right|^2 &\leqslant 256 \left|\partial_\tau \left(\tau \tilde{U}^a(a, 2\tau)\right)\right|^2 + 16 |E(a, \tau)|^2 \\ &+ 256 \int_{|x-a|=\tau} |\tilde{q}(x)|^2 \mathrm{d}\sigma(x) \int_{\mathrm{supp}\, \tilde{q} \cap |x-a|=\tau} |\tau k(x, \tau, a)|^2 \mathrm{d}\sigma(x) \\ &+ 256 \int_{|x-a|\leqslant\tau} |\tilde{q}(x)|^2 \mathrm{d}x \int_{\mathrm{supp}\, \tilde{q} \cap |x-a|\leqslant\tau} |\partial_\tau(\tau k(x, \tau, a))|^2 \mathrm{d}x\end{aligned}$$

Note that $q_1(x) = q_2(x) = 0$ for $|x-a| < h$. Also recall the estimates (24) and (25) for integrals of k from proposition 4.1. We can proceed then with

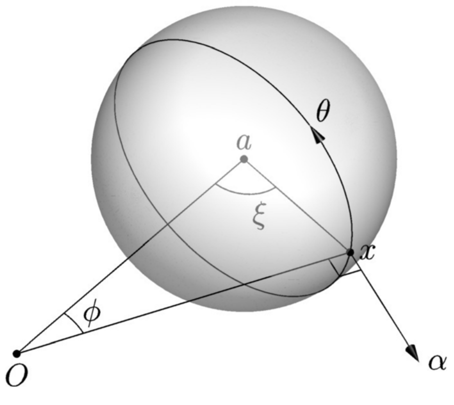

Figure 2. Reparametrization of $|x-a| = \tau$.

$$(1-\tau)^2 \left|\tilde{q}\big((1-\tau)a\big)\right|^2 \leqslant C_{M,h,B}\Big(\left|\partial_\tau \left(\tau \tilde{U}^a(a,2\tau)\right)\right|^2 + |E(a,\tau)|^2 + \int_{|x-a|=\tau} |\tilde{q}(x)|^2 \,\mathrm{d}\sigma(x) + \int_{|x-a|\leqslant\tau} |\tilde{q}(x)|^2 \,\mathrm{d}x \Big)$$

since $\|q_1\|_{C^7}, \|q_2\|_{C^7} \leqslant \mathcal{M}$.

Integrate the above estimate with $\int_{a\in\partial B} \ldots \mathrm{d}\sigma(a)$ and use the coordinate change of lemma 5.1. Then write $\mathcal{Q}(r) = \int_{|x|=r} |\tilde{q}(x)|^2 \,\mathrm{d}\sigma(x)$ and scale the integration variable on the left-hand side to get

$$\frac{\mathcal{Q}(1-\tau)}{C_{\mathcal{M},h,B}} \leqslant \int_{|a|=1} \left|\partial_\tau(\tilde{U}^a(a,2\tau))\right|^2 \mathrm{d}\sigma(a) + \int_{|a|=1} |E(a,\tau)|^2 \,\mathrm{d}\sigma(a) + \pi \int_{|x|\geqslant 1-\tau} |\tilde{q}(x)|^2 \frac{\tau^2 + 2\tau - (1-|x|)^2}{|x|} \mathrm{d}x. \tag{27}$$

Next, estimate $|E(a,\tau)|^2$ using proposition 4.2. Then change the order of integration using lemma 5.1, switch to angular coordinates, and apply angular control (3) to get

$$\begin{aligned}\int_{|a|=1} |E(a,\tau)|^2 \,\mathrm{d}\sigma(a) &\leqslant \frac{6\tau}{1-\tau} \sum_{i<j} \int_{|x|\geqslant 1-\tau} \frac{|\Omega_{ij}\tilde{q}(x)|}{|x|\sqrt{|x|-(1-\tau)}} \mathrm{d}\sigma(x) \\ &= \frac{6\tau}{1-\tau} \sum_{i<j} \int_{1-\tau}^{1} \int_{|x|=r} \frac{|\Omega_{ij}\tilde{q}(x)|}{r\sqrt{r-(1-\tau)}} \mathrm{d}\sigma(x)\mathrm{d}r \\ &\leqslant 6S^2 \int_{1-\tau}^{1} \frac{\tau}{1-\tau} \frac{\mathcal{Q}(r)}{r\sqrt{r-(1-\tau)}} \mathrm{d}r. \end{aligned} \tag{28}$$

Similarly, the last term in (27) can be written as

$$\ldots = \pi \int_{1-\tau}^{1} \frac{\tau^2 + 2\tau - (1-r)^2}{r} \mathcal{Q}(r)\mathrm{d}r. \tag{29}$$

Finally, combine estimates (28) and (29) to change (27) into

$$\mathcal{Q}(1-\tau) \leqslant C_{\mathcal{M},h,B} \int_{|a|=1} \left|\partial_\tau(\tau\tilde{U}^a(a,2\tau))\right|^2 \mathrm{d}\sigma(a) + C_{\mathcal{M},h,B} \int_{1-\tau}^{1} \left(\frac{6S^2\tau}{(1-\tau)r\sqrt{r-(1-\tau)}} + \pi\frac{\tau^2+2\tau-(1-r)^2}{r} \right) \mathcal{Q}(r)\mathrm{d}r$$

which is valid for $0 < \tau < 1$.

Our next step is to prepare for Grönwall's inequality. The inequality above can be written as

$$\varphi(\tau) \leqslant \mathrm{d}(\tau) + \int_0^\tau \beta(\tau,s)\varphi(s)\mathrm{d}s \tag{30}$$

for $0 < \tau < 1$ where

$$\varphi(\tau) = \mathcal{Q}(1-\tau), \qquad \mathrm{d}(\tau) = C_{\mathcal{M},h,B} \int_{|a|=1} \left|\partial_\tau(\tau\tilde{U}^a(a,2\tau))\right|^2 \mathrm{d}\sigma(a)$$

and

$$\beta(\tau,s) = C_{\mathcal{M},h,B} \left(\frac{6S^2\tau}{(1-\tau)(1-s)\sqrt{\tau-s}} + \pi\frac{\tau^2+2\tau-s^2}{1-s} \right).$$

Because of the singularities of β we restrict (30) to $0 < \tau \leqslant 1-\varepsilon$ for any given $\varepsilon > 0$. We have $1-s \geqslant 1-\tau \geqslant \varepsilon > 0$ and $\tau \leqslant 1$. In this situation we see easily that

$$\beta(\tau,s) \leqslant \frac{6C_{\mathcal{M},h,B}S^2}{\varepsilon^2\sqrt{\tau-s}} + \frac{3\pi C_{\mathcal{M},h,B}}{\sqrt{\varepsilon}\sqrt{\tau-s}} \leqslant \frac{6S^2+3\pi}{\varepsilon^2}\frac{C_{\mathcal{M},h,B}}{\sqrt{\tau-s}}.$$

Denote $C_{S,\mathcal{M},h,B} = (6S^2+3\pi)C_{\mathcal{M},h,B}$.

An application of Grönwall's inequality (lemma 5.2) implies

$$\varphi(\tau) \leqslant \left(1+2C_{S,\mathcal{M},h,B}\varepsilon^{-2}\right) \sup_{0<\tau_0<1} \mathrm{d}(\tau_0) \exp\left(4C^2_{S,\mathcal{M},h,B}\varepsilon^{-4}\tau\right) \tag{31}$$

for $0 < \tau \leqslant 1-\varepsilon$. Now, given any $\tau \in (0,1)$ we choose $\varepsilon > 0$ such that $\tau \leqslant 1-\varepsilon$ and the right-hand side of the estimate above is minimized. These conditions are satisfied for $\varepsilon = 1-\tau$. The claim (4) follows after recalling that $\varphi(\tau) = \int_{|x|=1-\tau} |(q_1-q_2)(x)|^2 \mathrm{d}\sigma(x)$ and applying simple estimates.

Let us prove the norm estimate for $\tilde{q} = q_1 - q_2$ over the whole B next. Rewrite (4) as

$$\|\tilde{q}\|_{L^2(\{|x|=r\})} \leqslant \Lambda \mathrm{e}^{\mathfrak{C}/r^4}$$

where $\Lambda = \|U_1^a - U_2^a\|$. Since $C_c^7(B) \hookrightarrow W^{1,\infty}(B)$ and the potentials are supported in B we have the Lipschitz-norm estimate $|\tilde{q}(x)| \leqslant \left|\tilde{q}(x+\ell\frac{x}{|x|})\right| + 2\ell\mathcal{M}$ for any $\ell \geqslant 0$. Integration gives

$$\|\tilde{q}\|_{L^2(\{|x|=r\})} \leqslant 2\sqrt{4\pi}\mathcal{M}r\ell + \frac{r}{r+\ell}\Lambda \mathrm{e}^{\mathfrak{C}/(r+\ell)^4}$$

which we can estimate to

$$\|\tilde{q}\|_{L^2(\{|x|=r\})} \leqslant 2\sqrt{4\pi}\mathcal{M}\ell + \Lambda \mathrm{e}^{\mathfrak{C}/\ell^4}$$

because $0 \leqslant r \leqslant 1$ and $\ell \geqslant 0$. The full domain estimate (5) follows from lemma 5.3.

The proof for $q_1 - q_2$ radially symmetric proceeds as above until (30). Since in the condition of angular control (3) we can assume that $S = 0$, we have

$$\beta(\tau, s) = C_{\mathcal{M},h,B}\pi\frac{\tau^2 + 2\tau - s^2}{1-s} \leqslant \frac{C'_{\mathcal{M},h,B}}{1-s}$$

and so

$$\frac{\varphi(\tau)}{C''_{\mathcal{M},h,B}} \leqslant \|U_1^a - U_2^a\|^2 + \int_0^\tau \frac{\varphi(s)}{1-s}\mathrm{d}s.$$

This type of integral inequality implies

$$\begin{aligned}\varphi(\tau) &\leqslant C''_{\mathcal{M},h,B}\|U_1^a - U_2^a\|^2 \exp\left(\int_0^\tau \frac{C''_{\mathcal{M},h,B}}{1-s}\mathrm{d}s\right)\\ &= C''_{\mathcal{M},h,B}\|U_1^a - U_2^a\|^2 (1-\tau)^{-2\alpha}\end{aligned}$$

for some $\alpha = \alpha(\mathcal{M}, h, B)$ by Grönwall's inequality. Note that here τ is allowed to be anywhere in the whole interval $(0, 1)$ without any of the constants blowing up. Following the rest of the proof implies Hölder stability. □

5. Technical tools

We collect here some basic calculations and some well known theorems so that we may refer to them without losing focus in the main proof.

Lemma 5.1. *Let f be a continuous function vanishing outside of B and let $\tau < 1$ positive. Then*

$$\int_{|a|=1}\int_{|x-a|=\tau} f(x)\mathrm{d}\sigma(x)\mathrm{d}\sigma(a) = 2\pi\tau\int_{|x|\geqslant 1-\tau}\frac{f(x)}{|x|}\mathrm{d}x$$

and

$$\int_{|a|=1}\int_{|x-a|\leqslant\tau} f(x)\mathrm{d}x\mathrm{d}\sigma(a) = \pi\int_{|x|\geqslant 1-\tau}\frac{f(x)}{|x|}\left(\tau^2 - (1-|x|)^2\right)\mathrm{d}x.$$

Proof. The first equation was proven just before formula (2.10) in [RU2]. The left-hand side of the second equation was shown to be equal to

$$\int_{|x|\leqslant 1} f(x)\int_{|a|=1} H(\tau^2 - |x-a|^2)\mathrm{d}\sigma(a)\mathrm{d}x$$

therein too.

The last equality follows by noting that the integral of the Heaviside function is just the area of the spherical cap arising from the intersection of $|a| = 1$ and $|a - x| = \tau$. If $|x| < 1 - \tau$ then this intersection is empty. Otherwise the area is seen to be $2\pi \cdot r \cdot h$, where $r = 1$ is the radius of the sphere $\{|a| = 1\}$ and h is the height of the cap along the ray $y\bar{0}$. Two applications of Pythagoras' theorem and some simple algebra imply that $h = (\tau^2 - (1 - |x|)^2)/(2\,|x|)$ and thus the final equality is proven. □

Lemma 5.2. *Let $b > a$ and $d\colon (a, b) \to \mathbb{R}$ be bounded and measurable. Moreover let $\beta\colon (\tau, s) \mapsto \beta(\tau, s)$ be measurable whenever $\tau, s \in (a, b)$ and $s < \tau$. Moreover let it satisfy*

$$\beta(\tau, s) \leqslant \frac{C}{\sqrt{\tau - s}}$$

for some $C < \infty$ whenever $s < \tau$.

If $\varphi\colon (a, b) \to \mathbb{R}$ is a non-negative integrable function that satisfies the integral inequality

$$\varphi(\tau) \leqslant \mathrm{d}(\tau) + \int_a^\tau \beta(\tau, s)\varphi(s)\mathrm{d}s \tag{32}$$

for almost all $\tau \in (a, b)$, then

$$\varphi(\tau) \leqslant (1 + 2C\sqrt{b - a}) \sup_{a < \tau_0 < b} \mathrm{d}(\tau_0)\mathrm{e}^{4C^2\tau}.$$

Proof. First of all note that since $\varphi \geqslant 0$, we may estimate β from above in the integral, and see that the former satisfies

$$\varphi(\tau) \leqslant \mathrm{d}(\tau) + C\int_a^\tau \frac{\varphi(s)}{\sqrt{\tau - s}}\mathrm{d}s$$

for almost all τ.

Next bootstrap the above by estimating φ inside the integral using that same inequality. Then

$$\varphi(\tau) \leqslant \mathrm{d}(\tau) + C\int_a^\tau \frac{\mathrm{d}(s)}{\sqrt{\tau - s}}\mathrm{d}s + C^2\int_a^\tau\int_a^s \frac{\varphi(s')}{\sqrt{\tau - s}\sqrt{s - s'}}\mathrm{d}s'\mathrm{d}s.$$

The double integral is estimated as follows: $\int_a^\tau\int_a^s \ldots \mathrm{d}s'\mathrm{d}s = \int_a^\tau\int_{s'}^\tau \ldots \mathrm{d}s\mathrm{d}s'$, and then we are left to estimate $\int_{s'}^\tau \mathrm{d}s/\sqrt{\tau - s}\sqrt{s - s'}$. To do that split the interval (s', τ) into two equal parts by the midpoint $s = (\tau + s')/2$. In the interval $s \in (s', (\tau + s')/2)$ we have $1/\sqrt{\tau - s} \leqslant \sqrt{2/(\tau - s')}$ and $\int_{s'}^{(\tau+s')/2} \mathrm{d}s/\sqrt{s - s'} = \sqrt{2(\tau - s')}$. Their product is equal to 2. The same deduction works in the second interval. Hence

$$\int_{s'}^\tau \frac{\mathrm{d}s}{\sqrt{\tau - s}\sqrt{s - s'}} \leqslant 4$$

indeed and

$$\varphi(\tau) \leqslant \mathrm{d}(\tau) + C\int_a^\tau \frac{\mathrm{d}(s)}{\sqrt{\tau - s}}\mathrm{d}s + 4C^2\int_a^\tau \varphi(s')\mathrm{d}s'$$

follows.

The first two terms above have an upper bound

$$(1 + 2C\sqrt{b-a}) \sup_{a<\tau_0<b} \mathrm{d}(\tau_0)$$

because $\int_a^\tau \mathrm{d}s/\sqrt{\tau - s} = 2\sqrt{\tau - a} \leqslant 2\sqrt{b-a}$. Grönwall's inequality implies the final claim: If $\varphi(\tau) \leqslant C_1 + C_2 \int_0^\tau \varphi(s)\mathrm{d}s$ for $\tau \geqslant 0$ where $\varphi \geqslant 0$ then $\varphi(\tau) \leqslant C_1 \exp(C_2\tau)$. This follows for example from appendix B.2.j in [Evans] and some algebra. Note however that the integral form of Grönwall's inequality in appendix B.2.k of [Evans] is weaker than this one. □

Lemma 5.3. *Let $f\colon \mathbb{R}_+ \to \mathbb{R}$ be a positive function satisfying*

$$f(\ell) \leqslant A\ell + \Lambda \mathrm{e}^{\mathfrak{C}/\ell^4}$$

for some $\Lambda < \infty$ and any ℓ in its domain. Then if $0 < \Lambda < e^{-1}$ we have

$$f(\ell_0) \leqslant \frac{A(2\mathfrak{C})^{1/4} + 2}{\left(\ln \frac{1}{\Lambda}\right)^{1/4}}$$

where $\ell_0^4 = \mathfrak{C}/(\ln \frac{1}{\sqrt{\Lambda}})$. If $\Lambda \geqslant e^{-1}$ then we have the linear estimate

$$f(\ell_0) \leqslant (A\mathfrak{C}^{1/4} + 1)e\Lambda.$$

for $\ell_0^4 = \mathfrak{C}$.

Proof. Since $\Lambda < e^{-1}$ the choice of ℓ_0 is proper. Moreover we see immediately that

$$f(\ell_0) \leqslant \frac{A(2\mathfrak{C})^{1/4}}{(\ln \frac{1}{\Lambda})^{1/4}} + \sqrt{\Lambda}.$$

Recall the elementary inequality $\ln \frac{1}{a} \leqslant \frac{1}{b} a^{-b}$ for $b > 0$ and $0 < a < e^{-1}$. Set $b = 2$ and $a = \Lambda$ to see that

$$\sqrt{\Lambda} \leqslant \frac{2}{\ln \frac{1}{\Lambda}} \leqslant \frac{2}{(\ln \frac{1}{\Lambda})^{1/4}}$$

since $\ln \frac{1}{\Lambda} > 1$ then. The first claim follows. The second claim is elementary. □

The following is from personal communication with Rakesh.

Lemma 5.4. *Let $p\colon \mathbb{R} \to \mathbb{R}$ be a measurable function. Then, given any time $t \geqslant 0$ and position $x \in \mathbb{R}^n$ with $t \geqslant |x|$, we have*

$$|y| + |x-y| \leqslant t \quad \Longleftrightarrow \quad (t - |y|)^2 - |x-y|^2 \geqslant 0$$

and

$$\int_{|y|+|x-y|\leqslant t} \frac{p\big((t-|y|)^2 - |x-y|^2\big)}{|y|} \mathrm{d}y = \int_{|w| \leqslant \frac{1}{2}\sqrt{t^2-|x|^2}} \frac{p\big((\sqrt{t^2-|x|^2} - |w|)^2 - |w|^2\big)}{|w|} \mathrm{d}w.$$

Proof. The first claim follows from the triangle inequality applied to a triangle with vertices x, y and $\bar{0}$: $t - |y| + |x-y| \geqslant |x| - |y| + |x-y| \geqslant 0$, so we may multiply the inequality

$$t - |y| - |x-y| \geqslant 0$$

by the former without changing sign.

Let $p_+(r) = p(r)$ for $r \geqslant 0$ and $p_+(r) = 0$ for $r < 0$. Denote the left-hand side integral in the statement by I. Then

$$\begin{aligned} I &= \int_{\mathbb{R}^3} \frac{p_+\big((t-|y|)^2 - |x-y|^2\big)}{|y|} \mathrm{d}y \\ &= \int_{\mathbb{R}^3} \int_{-\infty}^{\infty} \frac{\delta(s-|y|)}{|y|} p_+\big((t-|y|)^2 - |x-y|^2\big) \mathrm{d}s \mathrm{d}y \\ &= \int_{-\infty}^{\infty} \int_{\mathbb{R}^3} \frac{\delta(s-|y|)}{|y|} p_+\big((t-|y|)^2 - |x-y|^2\big) \mathrm{d}y \mathrm{d}s \\ &= 2\int_{-\infty}^{\infty} \int_{\mathbb{R}^3} \delta(s^2 - |y|^2)\, p_+\big((t-|y|)^2 - |x-y|^2\big) \mathrm{d}y \mathrm{d}s. \end{aligned}$$

Let $L_1 \colon \mathbb{R}^3 \to \mathbb{R}^3$ be a rotation taking $x \mapsto (|x|, 0, 0)$. Let it map $y \mapsto y'$. Then $\mathrm{d}y = \mathrm{d}y'$ and so

$$I = 2\int_{-\infty}^{\infty} \int_{\mathbb{R}^3} \delta(s^2 - |y'|^2)\, p_+\big((t-|y'|)^2 - |L_1 x - y'|^2\big) \mathrm{d}y' \mathrm{d}s.$$

Next let $(s, y') \mapsto z \in \mathbb{R}^4$ be the Lorentz transformation given by

$$z_0 = \frac{ts - |x|\, y_1'}{\sqrt{t^2 - |x|^2}}, \quad z_1 = \frac{ty_1' - |x|\, s}{\sqrt{t^2 - |x|^2}}, \quad z_2 = y_2, \quad z_3 = y_3.$$

It is a trivial matter to see that $\mathrm{d}z = \mathrm{d}y' \mathrm{d}s$ and the following identities

$$z_0^2 - z_1^2 = s^2 - y_1'^2, \qquad \left(\sqrt{t^2 - |x|} - z_0\right)^2 - z_1^2 = (t-s)^2 - (|x| - y_1')^2.$$

Finally, denoting $|z|^2 = z_1^2 + z_2^2 + z_3^2$ and $z \cdot z = z_0^2 - |z|^2$, we have

$$\begin{aligned} I &= 2\int_{\mathbb{R}^4} \delta(z \cdot z)\, p_+\big((\sqrt{t^2 - |x|^2} - z_0)^2 - |z|^2\big) \mathrm{d}z \\ &= \int_{-\infty}^{\infty} \int_{\mathbb{R}^3} \frac{\delta(z_0 - |z|)}{|z|} p_+\big((\sqrt{t^2 - |x|^2} - z_0)^2 - |z|^2\big) \mathrm{d}z_1 \mathrm{d}z_2 \mathrm{d}z_3 \mathrm{d}z_0 \\ &= \int_{\mathbb{R}^3} \frac{p_+\big((\sqrt{t^2 - |x|^2} - |z|)^2 - |z|^2\big)}{|z|} \mathrm{d}z_1 \mathrm{d}z_2 \mathrm{d}z_3 \\ &= \int_{\mathbb{R}^3} \frac{p_+\big((\sqrt{t^2 - |x|^2} - |w|)^2 - |w|^2\big)}{|w|} \mathrm{d}w \end{aligned}$$

which implies the claim since $(\sqrt{t^2 - |x|^2} - |w|)^2 - |w|^2 \geqslant 0$ if and only if $\sqrt{t^2 - |x|^2} - |w| - |w| \geqslant 0$. □

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References

[Bal1] Balean R 1996 The null-timelike boundary problem *PhD Thesis* University of New England
[Bal2] Balean R 1997 The null-timelike boundary problem for the linear wave equation *Commun. PDE* **22** 1325–60
[Cag] Cagnac F 1982 Problème de Cauchy sur un conoïde caractéristique pour des équations quasi-linéaires *Ann. Mat. Pura Appl.* **129** 13–41
[Evans] Evans L C 2010 Partial differential equations *Graduate Studies in Mathematics* vol 19, 2nd edn (Providence, RI: American Mathematical Society)
[Fri] Friedlander F G 1975 *The Wave Equation on a Curved Space-Time* (Cambridge: Cambridge University Press)
[Man] Mandache N 2001 Exponential instability in an inverse problem for the Schrödinger equation *Inverse Problems* **17** 1435–44
[MU] Melrose R and Uhlmann G 2008 Generalized backscattering and the Lax–Phillips transform *Serdica Math. J.* **34** 3557–72
[RU1] Rakesh and Uhlmann G 2014 Uniqueness for the inverse back-scattering problem for angularly controlled potentials *Inverse Problems* **30** 065005
[RU2] Rakesh and Uhlmann G 2015 The point source inverse back-scattering problem *Contemp. Math.* **644** 279–89
[Rom] Romanov V 1974 *Integral Geometry and Inverse Problems for Hyperbolic Equations* (Berlin: Springer)